ALGEBRA

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Problem 1. Given distinct integers $x_1, x_2, ..., x_n$, prove that $\prod_{i>j}(x_i - x_j)$ is divisible by $1!2!3!...(n-1)!$.

Problem 2. Prove that for any integers $x_1, x_2, ..., x_n$ and positive integers $k_1, k_2, ..., k_n$, the determinant

$$\det \begin{pmatrix} x_1^{k_1} & x_2^{k_1} & \cdots & x_n^{k_1} \\ x_1^{k_2} & x_2^{k_2} & \cdots & x_n^{k_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k_n} & x_2^{k_n} & \cdots & x_n^{k_n} \end{pmatrix}$$

is divisible by $n!$.

Problem 3. Let $A = (a_{ij})_{ij}$ be an $n \times n$ matrix such that $\sum_{j=1}^n |a_{ij}| < 1$ for each i. Prove that $I_n - A$ is invertible.

Problem 4. Let $\alpha = \frac{\pi}{n+1}, n > 2$. Prove that the $n \times n$ matrix

$$\begin{pmatrix} \sin\alpha & \sin 2\alpha & \cdots & \sin n\alpha \\ \sin 2\alpha & \sin 4\alpha & \cdots & \sin 2n\alpha \\ \vdots & \vdots & \ddots & \vdots \\ \sin n\alpha & \sin 2n\alpha & \cdots & \sin n^2\alpha \end{pmatrix}$$

is invertible.

Problem 5. Let A be an $n \times n$ such that there exists a positive integer k for which

$$kA^{k+1} = (k+1)A^k.$$

Prove that the matrix $A - I$ is invertible and find its inverse.

Problem 6. Let $P(x) = x^n + x^{n-1} + \cdots + x + 1$. Find the remainder obtained when $P(x^{n+1})$ is divided by $P(x)$.

Problem 7. Prove that every odd polynomial function of degree equal to $2m-1$ can be written as

$$P(x) = c_1\binom{x}{1} + c_2\binom{x+1}{3} + c_3\binom{x+2}{5} + \cdots + c_m\binom{x+m-1}{2m-1}$$

where $\binom{x}{m} = x(x-1)\cdots\frac{x-m+1}{m!}$.

Problem 8. Let n be a positive integer and $P(x)$ and nth-degree polynomial with complex coefficients such that $P(0), P(1), ..., P(n)$ are all integers. Prove that $n!P(x)$ has integer coefficients.

Problem 9. Let A be the $n \times n$ matrix whose i, j entry is $i + j$ for all $i, j = 1, 2, ..., n$. What is the rank of A?

Problem 10. There are given $2n+1$ real numbers, $n \geqslant 1$, with the property that whenever one of them is removed, the remaining $2n$ numbers can be split into two sets of n elements that have the same sum of elements. Prove that all of the numbers are equal.

Problem 11. Let A be an $n \times n$ matrix. Prove that there exists an $n \times n$ matrix B such that $ABA = A$.

Problem 12. A linear map A on the n-dimensional vector space V is called an involution if $A^2 = I$.

(1) Prove that for every involution A on V there exists a basis of V consisting of eigenvectors of A.
(2) Find the maximal number of distinct pairwise commuting involutions.

Problem 13. Let U and V be isometric linear transformations of $\mathbb{R}^n$, $n \geqslant 1$, with the property that $\|Ux - x\| \leqslant \frac{1}{2}$ and $\|Vx - x\| \leqslant \frac{1}{2}$ for all $x \in \mathbb{R}^n$ with $\|x\| = 1$. Prove that

$$\|UVU^{-1}V^{-1}x - x\| \leqslant \frac{1}{2},$$

for all $x \in \mathbb{R}^n$ with $\|x\| = 1$.

Problem 14. Let A be a square matrix whose off-diagonal entries are positive. Prove that the rightmost eigenvalue of A in the complex plane is real and all other eigenvalues are strictly to its left in the complex plane.

Problem 15. Let $x_1, x_2, ..., x_n$ be differentiable real-valued functions of a variable t that satisfy

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\cdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}$$

for some constants $a_{ij} > 0$. Suppose for all i that $x_i(t) \to 0$ as $t \to \infty$. Are the functions x_i linearly dependent?

Problem 16. Consider a set S and a binary operation $*$ on S such that $x * (y * x) = y$ for all $x, y \in S$. Prove that each of the equations $a * x = b$ and $x * a = b$ has a unique solution S.

Problem 17. On a set M an operation $*$ is given satisfying the properties

(1) There is $e \in M$ such that $x * e = x$ for all $x \in M$;
(2) $(x * y) * z = (z * x) * y$ for all $x, y, z \in M$.

Prove that the operation $*$ is both associative and commutative.