SEQUENCES AND SERIES

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Problem 1. Let $a = 4k - 1$, where k is an integer. Prove that for any positive integer n the number

$$1 - \binom{n}{2}a + \binom{n}{4}a^2 - \binom{n}{6}a^3 + ...$$

Problem 2. Prove that

$$\lim_{n\to\infty} n^2 \int_0^{\frac{1}{n}} x^{x+1} dx = \frac{1}{2}.$$

Problem 3. Let $(a_n)_n$ be a sequence of real numbers with the property that for any $n \geqslant 2$ there exists an integer k, $\frac{n}{2} \leqslant k < n$, such that $a_n = \frac{a_k}{2}$. Prove that $\lim_{n\to\infty} a_n = 0$.

Problem 4. Show that if the series $\sum a_n$ converges, where $(a_n)_n$ is a decreasing sequence, then $\lim_{n\to\infty} na_n = 0$.

Problem 5. Two maps of the same region drawn to different scales are superimposed so that the smaller map lies entirely inside the larger one. Prove that there is precisely one point on the small map that lies directly over a point on the large map that represents the same place of the region.

Problem 6. Consider the polynomial $P(x) = a_m x^m + a_{m-1}x^{m-1} + ... + a_0$, $a_i > 0$, $i = 0, 1, ..., m$. Denote by A_n and G_n the arithmetic and, respectively, geometric means of the numbers $P(1), P(2), ..., P(n)$. Prove that

$$\lim_{n\to\infty} \frac{A_n}{G_n} = \frac{e^m}{m+1}.$$

Problem 7. Let $f : [a, b] \to [a, b]$ be an increasing function. Show that there exists $\xi \in [a, b]$ such that $f(\xi) = \xi$.

Problem 8. Given a sequence $(a_n)_n$ such that for any $\gamma > 1$ the subsequence $a_{\lfloor \gamma^n \rfloor}$ converges to zero, does it follow that the sequence $(a_n)_n$ converges to 0?

Problem 9. Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function with the property that for any $x > 0$, $\lim_{n\to\infty} f(nx) = 0$. Prove that $\lim_{x\to\infty} f(x) = 0$.

Problem 10. Let $a_1, a_2, ..., a_n, ...$ be nonnegative numbers. Prove that $\sum_{n=1}^{\infty} a_n < \infty$ implies $\sum_{n=1}^{\infty} \sqrt{a_{n+1}a_n} < \infty$.

Problem 11. Does there exist a pair of divergent series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ with $a_1 \geqslant a_2 \geqslant a_3 \geqslant ... \geqslant 0$ and $b_1 \geqslant b_2 \geqslant b_3 \geqslant ... \geqslant 0$, such that $\sum_{n=1}^{\infty} \min(a_n, b_n)$ converges?

Problem 12. For a nonnegative integer k, define $S_k(n) = 1^k + 2^k + \cdots + n^k$. Prove that

$$1 + \sum_{k=0}^{r-1} \binom{r}{k} S_k(n) = (n+1)^r.$$

Problem 13. Observe that:

$$\frac{1}{1} = \frac{1}{2} + \frac{1}{2}$$
$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$
$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

etc. Using this or otherwise, prove that for any $n > 1$ there are i and j such that

$$\frac{1}{n} = \frac{1}{i(i+1)} + \frac{1}{(i+1)(i+2)} + \cdots + \frac{1}{j(j+1)}.$$

Problem 14. Let $h(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Prove for $n = 2, 3, 4, ...,$

$$n + h(1) + h(2) + ... + h(n-1) = nh(n).$$

Problem 15. A sequence $\{a_n\}$ of real numbers is defined by

$$a_1 = 1, \qquad a_{n+1} = 1 + a_1 a_2 ... a_n.$$

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = 2.$$

Problem 16. Find the sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{2n^2 - n}$.

Problem 17. a. Find a sequence (a_n), $a_n > 0$, such that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{1}{a_n}$$

both converge.

b. Prove that there is no sequence (a_n), $a_n > 0$, such that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{a_n}$$

both converge.

Problem 18. Find

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \right)^n,$$

or show that the limit does not exist.

Problem 19. If $\sum a_n$ converges, does there have to exist a periodic function $\epsilon : \mathbb{Z} \to \{1, -1\}$ such that $\sum \epsilon(n)|a_n|$ converges?

Problem 20. $\{A_n\}$ is a sequence of positive numbers satisfying $A_n < A_{n+1} + A_{n^2}$ for all n. Prove that $\sum A_n$ diverges.

Problem 21. The sum $\sum_{n=0}^{\infty} x^{n^2}$ tends to ∞ as $x \to 1^-$. How fast?

Problem 22. If $x_0 = 1$, $x_{n+1} = x_n + \frac{1}{x_n}$, then $x_n \to \infty$. How fast?

Problem 23. Given a convergent series of positive terms $\sum a_n$, prove $\sum (a_1 a_2 ... a_n)^{\frac{1}{n}}$ converges.

Problem 24. Prove that any sequence of real numbers contains a monotone subsequence.