NUMBER THEORY

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Problem 1. Three infinite arithmetic progressions are given, whose terms are positive integers. Assuming that each of the numbers $1, 2, 3, 4, 5, 6, 7, 8$ occurs in at least one of these progressions, show that 1980 occurs in one of them.

Problem 2. For a positive integer n and a real number x, compute the sum

$$\sum_{0 \leqslant i < j \leqslant n} \left\lfloor \frac{x+i}{j} \right\rfloor.$$

Problem 3. For p and q co-prime positive integers, prove

$$\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \left\lfloor \frac{3p}{q} \right\rfloor + \cdots + \left\lfloor \frac{(q-1)p}{q} \right\rfloor = \left\lfloor \frac{q}{p} \right\rfloor + \left\lfloor \frac{2q}{p} \right\rfloor + \left\lfloor \frac{3q}{p} \right\rfloor + \cdots + \left\lfloor \frac{(p-1)q}{p} \right\rfloor.$$

Problem 4. Let $p = 3k + 1$ be prime and let

$$\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(2k-1)2k} = \frac{m}{n}$$

for some coprime positive integers m and n. Prove that p divides m.

Problem 5. Let k and n be integers with $0 \leqslant k \leqslant \frac{n^2}{4}$. Assume that k has no prime divisor greater than n. Prove that $n!$ is divisible by k.

Problem 6. Prove that the sequence $2^n - 3$, $n \geqslant 1$, contains an infinite subsequence whose terms are pairwise relatively prime.

Problem 7. Let p be an odd prime and $a_1, a_2, ..., a_p$ an arithmetic progression whose common difference is not divisible by p. Prove that there exists an index i such that the number $a_1 a_2 ... a_p + a_i$ is divisible by p^2.

Problem 8. Prove for every positive integer n the identity

$$\phi(1) \left\lfloor \frac{n}{1} \right\rfloor + \phi(2) \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \phi(n) \left\lfloor \frac{n}{n} \right\rfloor = \frac{n(n+1)}{2}.$$

Problem 9. Is there a sequence of positive integers in which every positive integer occurs exactly once and for every $k = 1, 2, 3, ...$ the sum of the first k terms is divisible by k?

Problem 10. A lattice point $(x, y) \in \mathbb{Z}^2$ is visible from the origin if x and y are coprime. Prove that for any positive integer n there exists a lattice point (a, b) whose distance from every visible point is greater than n.

Problem 11. Prove that there exist infinitely many squares of form $1 + 2^{x^2} + 2^{y^2}$, where x and y are positive integers.

Problem 12. Prove that the equation $x^3 + y^3 + z^3 + t^3 = 1999$ has infinitely many integer solutions.

Problem 13. Calculate the sum

$$6 + 66 + \cdots + 666...6$$

in which the last summand has n digits that are 6's.

Problem 14. Let $a_1, a_2, ..., a_n$ be n positive integers. Show that for some i and k, $1 \leqslant i \leqslant i + k \leqslant n$

$$a_i + a_{i+1} + \cdots + a_{i+k}$$

is divisible by n.

Problem 15. Show that if m is a positive rational number then $m + \frac{1}{m}$ is an integer only if $m = 1$.

Problem 16. Prove that $1^{99} + 2^{99} + 3^{99} + 4^{99} + 5^{99}$ is divisible by 5.

Problem 17. Let f be a function with the following properties.

1. $f(n)$ is defined for every positive integer n
2. $f(n)$ is an integer
3. $f(2) = 2$
4. $f(mn) = f(m)f(n)$ all m, n
5. $f(m) > f(n)$ when $m > n$.

Prove that $f(n) = n$ for all n.

Problem 18. Prove that for $n = 1, 2, 3, ...,$

$$\left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n+4}{8} \right\rfloor + ... = n.$$

Problem 19. What is the maximum number of terms in a geometric progression with common ratio greater than 1 whose entries all come from the set of integers between 100 and 1000 inclusive?

Problem 20. Let n be a positive integer. Prove that the binomial coefficients

$$\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}$$

are all even if and only if n is a power of 2.

Problem 21. Prove that, for any positive integer n,

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+2}],$$

where [.] denotes the greatest integer function.

Problem 22. Prove that any positive integer n has a representation

$$n = \epsilon_1 1^2 + \epsilon_2 2^2 + \cdots + \epsilon_m m^2$$

where each $\epsilon_i = \pm 1$.

Problem 23. Suppose a and b are distinct real numbers such that

$$a - b, a^2 - b^2, a^3 - b^3, ...$$

are all integers.

(1) Must a and b be rational?
(2) Must a and b be integers?

Problem 24. On the multiplication table of numbers 1 through n times numbers 1 through n, show that the fraction of numbers 1 through n^2 which appear tends to 0.

Problem 25. Consider the sequence $1, 2, 3, 4, 6, 8, 9, 12, 16, 18, ...$ of positive numbers composed of 2's and 3's, arranged in increasing order. Prove that the ratio of consecutive terms converges to 1.

Problem 26. #44 Prove that the integers $\lfloor(\sqrt{2}+1)^n\rfloor$ are alternatively even and odd.

Problem 27. Let $[\alpha, \beta]$ be an interval which contains no integers. Prove that there is an integer n so that $[n\alpha, n\beta]$ contains no integers and has length at least $\frac{1}{6}$.

Problem 28. Prove that the product of three consecutive integers is never a perfect power.

Problem 29. How many perfect squares are there mod 2^n.

Problem 30. If a set of positive integers has sum n, what is the biggest its product can be?