# MULTIVARIABLE CALCULUS

ROBERT HOUGH

*Problem* 1. Let $0 < a < b$ and $t_i \geqslant 0$, $i = 1, 2, ..., n$. Prove that for any $x_1, x_2, ..., x_n \in [a, b]$,

$$\left(\sum_{i=1}^{n} t_i x_i\right)\left(\sum_{i=1}^{n} \frac{t_i}{x_i}\right) \leqslant \frac{(a+b)^2}{4ab}\left(\sum_{i=1}^{n} t_i\right)^2.$$

*Problem* 2. Let $0 < x_i < \pi$, $i = 1, 2, ..., n$, and set $x = \frac{x_1 + x_2 + \cdots + x_n}{n}$. Prove that

$$\prod_{i=1}^{n}\left(\frac{\sin x_i}{x_i}\right) \leqslant \left(\frac{\sin x}{x}\right)^n.$$

*Problem* 3. Compute

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx.$$

Calculate

$$\lim_{n\to\infty}\left[\frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n-1)}\right]\cdot\frac{1}{n} = \pi.$$

*Problem* 4. Compute

$$\lim_{n\to\infty}\left(\frac{2^{\frac{1}{n}}}{n+1} + \frac{2^{\frac{2}{n}}}{n+\frac{1}{2}} + \cdots + \frac{2^{\frac{n}{n}}}{n+\frac{1}{n}}\right).$$

*Problem* 5. Prove that for any real $x$ the series

$$1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \frac{x^{12}}{12!} + ...$$

is convergent and find its limit.

*Problem* 6. Prove that if the function $u(x, t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad (x, t) \in \mathbb{R}^2,$$

then so does the function

$$v(x, t) = \frac{1}{\sqrt{t}} u(xt^{-1}, -t^{-1}), \qquad x \in \mathbb{R}, t > 0.$$

*Problem* 7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function with continuous partial derivatives and with $f(0, 0) = 0$. Prove that there exist continuous functions $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x, y) = xg_1(x, y) + yg_2(x, y).$$

*Problem* 8. Calculate the integral of the function

$$f(x, y, z) = \frac{x^4 + 2y^4}{x^4 + 4y^4 + z^4}$$

over the ball $B = \{(x, y, z) : x^2 + y^2 + z^2 \leqslant 1\}$.

*Problem* 9. Assume that a curve $(x(t), y(t))$ runs counterclockwise around a region $D$. Prove that the area of $D$ is given by the formula

$$A = \frac{1}{2} \oint_{\partial D} (xy' - yx')dt.$$

*Problem* 10. Compute

$$\oint_C y^2 dx + z^2 dy + x^2 dz,$$

where $C$ is the Viviani curve, defined as the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ with the cylinder $x^2 + y^2 = ax$.

*Problem* 11. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be twice continuously differentiable functions that are constant along lines that pass through the origin. Prove that on the unit ball $B = \{(x, y, z) : x^2 + y^2 + z^2 \leqslant 1\}$,

$$\iiint_B f \Delta g dV = \iint_B g \Delta f dV$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian.

*Problem* 12. Prove Gauss's Law, which states that the total flux of a gravitational field through a closed surface equals $-4\pi G$ times the mass enclosed by the surface, where $G$ is the gravitational constant.

*Problem* 13. Let $n$ be a positive integer. Show that the equation

$$(1 - x^2)y'' - xy' + n^2 y = 0$$

admits as a particular solution an $n$th degree polynomial.

*Problem* 14. The function $f$ has the property that

$$|f(a) - f(b)| \leqslant |a - b|^2$$

for all real $a, b$. Show that $f$ is a constant.

*Problem* 15. If $a_0 \geqslant a_1 \geqslant a_2 \geqslant ... \geqslant a_n > 0$, prove that any root $r$ of the polynomial

$$P(z) \equiv a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

satisfies $|r| \leqslant 1$.

*Problem* 16. Does there exist a continuous function $y = f(x)$, defined for all real $x$, whose graph intersects every non-vertical line in infinitely many points?

*Problem* 17. Is there a function $f$, differentiable for all real $x$, such that

$$|f(x)| < 2, \qquad f(x)f'(x) \geqslant \sin x?$$

*Problem* 18. Does the Maclaurin series for $e^{x-x^3}$ have any zero coefficients?

*Problem* 19. A function $f(x) \in C^\infty[0, \infty)$ is called *completely monotonic* if $(-\frac{d}{dx})^k f(x) \geqslant 0$ for all $k = 0, 1, 2, ...$ and all $x \geqslant 0$. These functions form a convex cone. Prove that the functions $\alpha e^{-\beta x}$, $\alpha, \beta \geqslant 0$ are extreme points.

*Problem* 20. A function $f(x) \in C[0, \infty)$ is called *slow* if $f(x + a) - f(x) \to 0$ as $x \to \infty$ for each fixed $a$. Prove that a slow function can be written as the sum $g(x) + h(x)$, where $g(x) \to 0$ and $h'(x) \to 0$ as $x \to \infty$.

*Problem* 21. $f(x)$ is continuous on $[0, \infty)$, and is such that, for each fixed $a > 0$, $f(na) \to 0$. Must $f(x) \to 0$ as $x \to \infty$?

*Problem* 22. Show that $f(x) \in C^1[a, b]$ iff the limit as $h \to 0$ of $(f(x+h) - f(x))/h$ exists uniformly on $[a, b]$.

*Problem* 23. Let $f(x) \in C^2$. Show that if $f(x)$ and $f''(x)$ are bounded, then $f'(x)$ is.

*Problem* 24. Given that $f(x) + f'(x) \to 0$ as $x \to \infty$, prove that both $f(x) \to 0$ and $f'(x) \to 0$.

*Problem* 25. $\sin \sin \sin \sin ... \sin(\pi/2)$ ($n$ interates) approaches 0 as $n \to \infty$. Obtain a rate.