# LINEAR ALGEBRA

ROBERT HOUGH

*Problem* 1. Let $a, b, c, d$ be real numbers such that $c \neq 0$ and $ad - bc = 1$. Prove that there exist $u$ and $v$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix}.$$

*Problem* 2. Calculate the $n$th power of the $m \times m$ matrix

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

*Problem* 3. Derive the formula for the determinant of a circulant matrix

$$\det \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_3 & x_4 & x_5 & \cdots & x_2 \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{pmatrix} = (-1)^{n-1} \prod_{j=0}^{n-1} \left( \sum_{k=1}^{n} \zeta^{jk} x_k \right),$$

$\zeta = e^{2\pi i/n}$.

*Problem* 4. Compute the determinant of the $n \times n$ matrix $A = (a_{ij})_{ij}$ where $a_{ij} = (-1)^{|i-j|}$ if $i \neq j$ and $a_{ii} = 2$.

*Problem* 5. Prove that for any integers $x_1, x_2, ..., x_n$ and positive integers $k_1, k_2, ..., k_n$, the determinant

$$\det \begin{pmatrix} x_1^{k_1} & x_2^{k_1} & \cdots & x_n^{k_1} \\ x_1^{k_2} & x_2^{k_2} & \cdots & x_n^{k_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k_n} & x_2^{k_n} & \cdots & x_n^{k_n} \end{pmatrix}$$

is divisible by $n!$.

*Problem* 6. Let $P(t)$ be a polynomial of even degree with real coefficients. Prove that $f(X) = P(X)$ defined on the set of $n \times n$ matrices is not onto.

*Problem* 7. Let $A = (a_{ij})_{ij}$ be an $n \times n$ such that $\sum_{j=1}^{n} |a_{ij}| < 1$ for each $i$. Prove that $I - A$ is invertible.

*Problem* 8. Let $A$ be an $n \times n$ matrix such that there exists a positive integer $k$ for which $kA^{k+1} = (k+1)A^k$. Prove that $A - I$ is invertible and find its inverse.

*Problem* 9. A linear map $A$ on the $n$-dimensional vector space $V$ is called an involution if $A^2 = I$.

a. Prove that for every involution $A$ on $V$ there exists a basis of $V$ consisting of eigenvectors of $A$.
b. Find the maximal number of distinct pairwise commuting involutions.

*Problem* 10. Find the $2 \times 2$ matrices with real entries that satisfy the equation

$$X^3 - 3X^2 = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}.$$

*Problem* 11. Let $x_1, x_2, ..., x_n$ be differentiable (real-valued) functions of a single variable $t$ that satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n,$$
$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n,$$
$$\cdots$$
$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n,$$

for constants $a_{ij} > 0$. Suppose for all $i$ that $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_i$ necessarily linearly dependent?

*Problem* 12. Let $A$ be a $4 \times 4$ matrix such that each entry of $A$ is either 2 or $-1$. Let $d = \det(A)$. Show that $d$ is divisible by 27.

*Problem* 13. For any vector $v$ in $\mathbb{R}^n$ and permutation $\sigma$ of $\{1, 2, ..., n\}$, define $\sigma(v) = (x_{\sigma(1)}, ..., x_{\sigma(n)})$. What are the possibilities for the dimension of the space spanned by $\sigma(v)$ such that $\sigma$ is a permutation?

*Problem* 14. Let $f_1, f_2, ..., f_n$ be linearly independent, differentiable functions. Prove that some $n - 1$ of their derivatives $f_1', f_2', ..., f_n'$ are independent.