INEQUALITIES

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Problem 1. Let $a_1, a_2, ..., a_n$ be positive real numbers with $a_1 + a_2 + ... + a_n < 1$. Prove that

$$\frac{a_1 a_2 ... a_n (1 - a_1 - a_2 - ... - a_n)}{(a_1 + a_2 + ... + a_n)(1 - a_1)(1 - a_2)...(1 - a_n)} \leqslant \frac{1}{n^{n+1}}.$$

Problem 2. Consider the positive real numbers $x_1, x_2, ..., x_n$ with $x_1 x_2 ... x_n = 1$. Prove that

$$\frac{1}{n - 1 + x_1} + \frac{1}{n - 1 + x_2} + ... + \frac{1}{n - 1 + x_n} \leqslant 1.$$

Problem 3. Show that for all real u_j, v_j and x_j, $1 \leqslant j \leqslant n$, one has the following upper bound for a product of two linear forms:

$$\sum_{j=1}^n u_j x_j \sum_{j=1}^n v_j x_j \leqslant \frac{1}{2} \left(\sum_{j=1}^n u_j v_j + \left(\sum_{j=1}^n u_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n v_j^2 \right)^{\frac{1}{2}} \right) \sum_{j=1}^n x_j^2.$$

Problem 4. Given $\mathbf{x} = (t; x_1, x_2, ..., x_d)$ and $\mathbf{y} = (u; y_1, y_2, ..., y_d)$, their Lorentz product is

$$[\mathbf{x}, \mathbf{y}] = tu - x_1 y_1 - x_2 y_2 - ... - x_d y_d.$$

The light cone is $\sqrt{x_1^2 + ... + x_d^2} \leqslant t$. Prove that if $\mathbf{x}$ and $\mathbf{y}$ are in the light cone, then

$$[\mathbf{x}, \mathbf{x}]^{\frac{1}{2}} [\mathbf{y}, \mathbf{y}]^{\frac{1}{2}} \leqslant [\mathbf{x}, \mathbf{y}].$$

Problem 5. Consider real weights $p_j > 0$, $j = 1, 2, ..., n$ arbitrary real numbers α_j, $j = 1, 2, ..., n$, and an inner product space $(V, \langle \cdot, \cdot, \rangle)$. Prove

$$\left\| \sum_{j=1}^n p_j \alpha_j \mathbf{x}_j \right\|^2 \leqslant \sum_{j=1}^n p_j \alpha_j^2 \sum_{k=1}^n p_k \|\mathbf{x}_k\|^2$$

for all $\mathbf{x}_k$, $1 \leqslant k \leqslant n$, in V.

Problem 6. Let $\mathbf{u}, \mathbf{v}$ be elements of a real inner product space satisfying

$$\langle \mathbf{u}, \mathbf{u} \rangle \leqslant A, \qquad \langle \mathbf{v}, \mathbf{v} \rangle \leqslant B.$$

Prove

$$\left(A^2 - \langle \mathbf{u}, \mathbf{u} \rangle\right)^{\frac{1}{2}} \left(B^2 - \langle \mathbf{v}, \mathbf{v} \rangle\right)^{\frac{1}{2}} \leqslant AB - \langle \mathbf{u}, \mathbf{v} \rangle.$$

Problem 7. Show that if $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx \leqslant 2 \left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |f'(x)|^2 dx \right)^{\frac{1}{2}}.$$

Problem 8. Prove, for $x > 0$, that

$$\int_x^{\infty} e^{-u^2/2} du \leqslant \frac{1}{x} e^{-x^2/2}.$$

Problem 9. Suppose that $-1 \leqslant x_1 < x_2 < ... < x_n \leqslant 1$, and show that

$$\sum_{1\leqslant j<k\leqslant n} \frac{1}{x_k - x_j} \geqslant \frac{1}{8} n^2 \log n.$$

Problem 10. For a polynomial $P(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ with real or complex coefficients, the smallest $r(P)$ such that all roots of P are contained in the disk $\{z : |z| \leqslant r(P)\}$ is called the *inclusion radius* for P. Show that for $p > 1$ and $q = \frac{p}{p-1} > 1$ one has the bound

$$r(P) < (1 + A_p^q)^{\frac{1}{q}}, \qquad A_p = \left(\sum_{j=0}^{n-1} |a_j|^p\right)^{\frac{1}{p}}.$$

Problem 11. If $\phi : [0, \infty) \to [0, \infty)$ is an integrable function and $t \in (0, \infty)$, then the integral

$$\mu_t = \int_0^\infty x^t \phi(x) dx$$

is called the tth moment of ϕ. Show that if $t \in (t_0, t_1)$ then

$$\mu_t \leqslant \mu_{t_0}^{1-\alpha} \mu_{t_1}^{\alpha}, \qquad t = (1-\alpha)t_0 + \alpha t_1.$$

Problem 12. Let $1 \leqslant s_0, t_0, s_1, t_1 \leqslant \infty$ be given and consider an $m \times n$ matrix T with nonnegative real entries c_{jk}, $1 \leqslant j \leqslant m$, $1 \leqslant k \leqslant n$. Suppose there exist constants M_0 and M_1 such that

$$\|T\mathbf{x}\|_{t_0} \leqslant M_0 \|\mathbf{x}\|_{s_0}, \qquad \|T\mathbf{x}\|_{t_1} \leqslant M_1 \|\mathbf{x}\|_{s_1},$$

for all $\mathbf{x} \in \mathbb{R}^m$, then for any $0 \leqslant \theta \leqslant 1$, one has the bound

$$\|T\mathbf{x}\|_t \leqslant M_\theta \|\mathbf{x}\|_s$$

where $M_\theta = M_1^\theta M_0^{1-\theta}$ and where s and t are given by

$$\frac{1}{s} = \frac{\theta}{s_1} + \frac{1-\theta}{s_0}, \qquad \frac{1}{t} = \frac{\theta}{t_1} + \frac{1-\theta}{t_0}.$$

Problem 13. Given an $n \times n$ array of positive numbers

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}$$

let m_j be the minimum of the jth column and m the max of the m_j's. Let M_i be the max of the ith row and let M be the min of the M_i's. Prove $m \leqslant M$.

Problem 14. Suppose that for $0 \leqslant x_i \leqslant 1$ for $i = 1, 2, ..., n$. Prove that

$$2^{n-1}(1 + x_1 x_2 ... x_n) \geqslant (1 + x_1)(1 + x_2)...(1 + x_n).$$

Problem 15. Suppose x, y, z are non-negative real numbers. Prove that

$$8(x^3 + y^3 + z^3) \geqslant 9(x^2 + yz)(y^2 + xz)(z^2 + xy).$$

Problem 16. If a, b, c are the lengths of the sides of a triangle, prove that

$$abc \geqslant (a + b - c)(b + c - a)(c + a - b).$$

Problem 17. Consider the $n \times n$ array whose entry in the ith row, jth column is $i + j - 1$. What is the smallest product of n numbers from this array, with one from each row and one from each column?