GEOMETRY AND TRIGONOMETRY

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Problem 1. There are n line segments in the plane with the sum of lengths equal to 1. Prove that there exists a straight line such that the sum of the lengths of the projections of the segments onto the line is equal to $\frac{2}{\pi}$.

Problem 2. Let $a_0, a_1, ..., a_n$ be numbers from the interval $(0, \frac{\pi}{2})$ such that

$$\tan\left(a_0 - \frac{\pi}{4}\right) + \tan\left(a_1 - \frac{\pi}{4}\right) + \cdots + \tan\left(a_n - \frac{\pi}{4}\right) \geqslant n - 1.$$

Prove that $\tan a_0 \tan a_1 ... \tan a_n \geqslant n^{n+1}$.

Problem 3. Prove the identity

$$\left(\frac{1 + i\tan t}{1 - i\tan t}\right)^n = \frac{1 + i\tan nt}{1 - i\tan nt}, \qquad n \geqslant 1.$$

Problem 4. Compute the sum

$$\binom{n}{1}\cos x + \binom{n}{2}\cos 2x + ... + \binom{n}{n}\cos nx.$$

Problem 5. Find

$$(\cos\alpha)(\cos 2\alpha)(\cos 3\alpha)\cdots(\cos 999\alpha)$$

with $\alpha = \frac{2\pi}{1999}$.

Problem 6. Prove that

$$\frac{1}{\sin 45 \sin 46} + \frac{1}{\sin 47 \sin 48} + \cdots + \frac{1}{\sin 133 \sin 134} = \frac{1}{\sin 1}.$$

Problem 7. Obtain explicit values of the following series:

a $\sum_{n=1}^{\infty} \arctan \frac{2}{n^2}$
b $\sum_{n=1}^{\infty} \arctan \frac{8n}{n^4 - 2n^2 + 5}$.

Problem 8. Compute the product

$$(\sqrt{3} + \tan 1)(\sqrt{3} + \tan 2)\cdots(\sqrt{3} + \tan 29).$$

Problem 9. Prove the identities

$$\left(\frac{1}{2} - \cos\frac{\pi}{7}\right)\left(\frac{1}{2} - \cos\frac{3\pi}{7}\right)\left(\frac{1}{2} - \cos\frac{9\pi}{7}\right) = -\frac{1}{8}$$

and

$$\left(\frac{1}{2} + \cos\frac{\pi}{20}\right)\left(\frac{1}{2} + \cos\frac{3\pi}{20}\right)\left(\frac{1}{2} + \cos\frac{9\pi}{20}\right)\left(\frac{1}{2} + \cos\frac{27\pi}{20}\right) = \frac{1}{16}.$$

Problem 10. Two points on a sphere of radius 1 are joined by an arc of length less than 2, lying inside the sphere. Prove that the arc is contained in some hemisphere.

Problem 11. Prove that if 5 pins are stuck onto a piece of cardboard in the shape of an equilateral triangle of side length 2, then some pair of pins must be at within distance 1 of each other.

Problem 12. Let a, b, and c be the lengths of the sides of a triangle. Show that if $a^2 + b^2 + c^2 = bc + ca + ab$ then the triangle is equilateral.

Problem 13. Given n points in the plane, any ordering of them $p_1, ..., p_n$ determines a path of straight line segments p_1 to p_2, p_2 to p_3, ..., p_{n-1} to p_n. Prove that the shortest such path does not cross itself.

Problem 14. $2n + 3$ points are given in the plane, no three on a line and no four on a circle. Prove that there exists a circle through three of them, such that, of the remaining points, n are in the interior of the circle and n on the exterior.

Problem 15. Prove that the sum of the areas of any three faces of a tetrahedron is greater than the area of the fourth face.

Problem 16. Show that for $k = 1, 2, 3, ...$

$$\sin \frac{\pi}{2k} \sin \frac{3\pi}{2k} \sin \frac{5\pi}{2k} \cdots \sin \left(2 \left\lfloor \frac{k+1}{2} \right\rfloor - 1\right) \frac{\pi}{2k} = \frac{1}{2^{\frac{k-1}{2}}}.$$

Problem 17. Show that any polygon can be tiled by convex pentagons.

Problem 18. Given n points in the unit square, there is a shortest curve connecting them. Estimate the length of this curve.

Problem 19. Given a right triangle and a finite set of points inside it, prove that these points can be connected by a path of line segments the sum of whose squares is bounded by the square of the hypotenuse.

Problem 20. Prove that there is no equilateral triangle all of whose vertices are plane lattice points.