INVARIANTS AND SEMI-INVARIANTS, ALGEBRAIC IDENTITIES

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In invariant of a system is a quantity which doesn't change as the system evolves, like the total energy or momentum of a classical system. A semi-invariant can change as the system evolves, but only increases or decreases. Consider the following example.

Example 1. Suppose n markers are given in a row, each with one white side and one black side. A move consists of selecting a white marker, not one of the outermost ones, removing it from the row, and flipping the color of its two neighbors. Prove that it is possible to reach a configuration with only two remaining markers if and only if $n-1$ is not divisible by 3.

Proof. The number of black markers is always even. Hence if two markers remain, either both are white or both are black. For each white marker, let t be the number of black markers to its left, and assign the marker the value $(-1)^t$. Let S be the sum of the values of the white markers. The value of S modulo 3 is an invariant, as can be checked by checking the 4 possible arrangments of the adjacent markers (black, white on either side). Since the initial configuration has S of value n mod 3, and the final configuration has $S \equiv 0$ or 2 mod 3 the claim is necessary. To prove sufficiency, check that the solution is possible if there are 3 or 5 white markers. If there are $n > 5$ white markers initially, successively take away the left-most available white marker 3 times. This leaves $n-3$ white markers. □

Example 2. Given a triple of numbers, select two of them a, b and replace them with $\frac{a+b}{\sqrt{2}}, \frac{a-b}{\sqrt{2}}$. Is it possible to change $(1, \sqrt{2}, 1+\sqrt{2})$ to $(2, \sqrt{2}, \frac{1}{\sqrt{2}})$?

Proof. The sum of the squares of the numbers is an invariant. This proves that the change is impossible. □

Example 3. Let x, y, z be distinct real numbers. Prove that

$$(x-y)^{\frac{1}{3}} + (y-z)^{\frac{1}{3}} + (z-x)^{\frac{1}{3}} \neq 0.$$

Proof. Suppose the contrary. Put $a = (x-y)^{\frac{1}{3}}$, $b = (y-z)^{\frac{1}{3}}$, $c = (z-x)^{\frac{1}{3}}$. From the identity

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca),$$

it follows that $a^3 + b^3 + c^3 = 3abc$, so $abc = 0$, contradiction. □

Problem 1. Starting with an ordered quadruple of positive integers, a generalized Euclidean algorithm is applied successively as follows: if the numbers are x, y, u, v and $x > y$ then the quadruple is replaced by $x-y, y, u+v, v$. Otherwise, it is replaced by $x, y-x, u, v+u$. The algorithm stops when the numbers in the first pair become equal. Assume that we start with m, n, m, n. Prove that when the algorithm ends the arithmetic mean of the last two numbers is the least common multiple of m and n.

Problem 2. Four congruent right triangles are given. One can cut one of them along the altitude, and repeat the operation several times with the newly obtained triangles. Prove that, no matter how the cuts are performed, we can always find among the triangles a pair that are congruent.

Problem 3. Starting with an ordered quadruple of integers, perform repeatedly the operation

$$(a,b,c,d) \mapsto (|a-b|,|b-c|,|c-d|,|d-a|).$$

Prove that after finitely many steps, the quadruple becomes $(0,0,0,0)$.

Problem 4. Show that for no positive integer n are both $n+3$ and n^2+3n+3 perfect cubes.

Problem 5. Prove that any polynomial which takes only non-negative values can be written as the sum of the squares of two polynomials.

Problem 6. Prove that for any odd integer $n \geqslant 5$,

$$\binom{n}{0}5^{n-1} - \binom{n}{1}5^{n-2} + \binom{n}{2}5^{n-2} - \cdots + \binom{n}{n-1}$$

is not prime.

Problem 7. Prove that, for infinitely many natural numbers a, n^4+a is not prime for any natural number n.

Problem 8. Factor $5^{1985}-1$ into a product of 3 integers, each of which is greater than 5^{100}.

Problem 9. Solve $(x-1)^{\frac{1}{3}} + x^{\frac{1}{3}} + (x+1)^{\frac{1}{3}} = 0$.

Problem 10. Suppose that n is the sum of two triangular numbers,

$$n = \frac{a^2+a}{2} + \frac{b^2+b}{2}.$$

Show that $4n+1$ is the sum of two squares. Conversely, show that if $4n+1$ is the sum of two squares, n is the sum of two triangular numbers.

Problem 11.

a. If a and b are consecutive integers, show that $a^2+b^2+(ab)^2$ is a perfect square.
b. If $a = \frac{b+c}{bc}$ then $a^2+b^2+c^2$ is the square of a rational number.
c. If N differs from the two consecutive squares between which it lies by x and y, respectively, prove that $N-xy$ is a square.

Problem 12. Let there be given nine lattice points in three-dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.

Problem 13. Remove the lower left corner square and the upper right corner square from an 8 by 8 chess board. Can the board be covered by 2 by 1 dominoes?

Problem 14.

a. Exploit symmetry to expand the product

$$(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2).$$

b. If $x+y+z=0$, prove that

$$\left(\frac{x^2+y^2+z^2}{2}\right)\left(\frac{x^5+y^5+z^5}{5}\right) = \frac{x^7+y^7+z^7}{7}.$$

Problem 15. Verify that the product of four consecutive terms of an arithmetic progression plus the fourth power of the common difference is always a perfect square.

Problem 16. Given a sequence of integers $x_1, x_2, ..., x_n$ whose sum is 1, prove that exactly one of the cyclic shifts

$$x_1, x_2, ..., x_n; \quad x_2, x_3, ..., x_n, x_1; \quad ...; \qquad x_n, x_1, ..., x_{n-1}$$

has all of its partial sums positive.

Problem 17. Show that if $a_1, ..., a_n$ are non-negative numbers, then

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geqslant (1 + (a_1...a_n)^{\frac{1}{n}})^n.$$

Problem 18. Let $x_1, x_2, ..., x_k$ be real numbers such that the set $A = \{\cos(n\pi x_1) + \cos(n\pi x_2) + \cdots + \cos(n\pi x_k) | n \geqslant 1\}$ is finite. Prove that the x_i are all rational.

Problem 19. The positive integers are colored by two colors. Prove that there exists an infinite sequence of positive integers $k_1 < k_2 < k_3 < ...$ with the property that $2k_1 < k_1 + k_2 < 2k_2 < k_2 + k_3 < ...$ all have the same color.