INEQUALITIES

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Among the most frequently used inequalities is *Cauchy-Schwarz.* If v, w are vectors in an inner product space,

$$|\langle v, w\rangle| \leqslant \|v\|\|w\|$$

with equality if and only if v and w are colinear. In $\mathbb{R}^n$, the inequality means that

$$|x_1y_1 + \cdots + x_ny_n| \leqslant (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}(y_1^2 + \cdots + y_n^2)^{\frac{1}{2}}.$$

Example 1. We have the power series inequality, for $|x| < 1$, and $\sum a_k^2 < \infty$,

$$\left|\sum_{k=0}^{\infty} a_k x^k\right| \leqslant \frac{1}{\sqrt{1-x^2}}\left(\sum_{k=0}^{\infty} a_k^2\right)^{\frac{1}{2}}.$$

Proof. The sequences $(x^k)_{k\geqslant 0}$ and $(a_k)_{k\geqslant 0}$ are in ℓ^2, and hence

$$\begin{aligned}|\langle (x^k), (a_k)\rangle| &= \left|\sum_k a_k x^k\right| \\ &\leqslant \|(x^k)\|_2\|(a_k)\|_2 \\ &= \left(\sum_k x^{2k}\right)^{\frac{1}{2}}\left(\sum_k a_k^2\right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{1-x^2}}\left(\sum_k a_k^2\right)^{\frac{1}{2}}.\end{aligned}$$

□

Another commonly used inequality is *Jensen's inequality.* Let $f : [a, b] \to \mathbb{R}$ be convex, in the sense that, for $x, y \in [a, b]$, $f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}$. Then if $0 \leqslant p_1, ..., p_n$ and $p_1 + \cdots + p_n = 1$, for any $x_1, ..., x_n \in [a, b]$,

$$f\left(\sum_j p_j x_j\right) \leqslant \sum_j p_j f(x_j).$$

Jensen's inequality implies many of the classical inequalities.

Example 2. Jensen's inequality implies Hölder's inequality. For $p \geqslant 1$, let $\|x\|_p = \left(\sum_j |x_j|^p\right)^{\frac{1}{p}}$. Let $q = \frac{p}{p-1}$ be the conjugate exponent, so $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left|\sum x_j y_j\right| \leqslant \|x\|_p\|y\|_q.$$

Proof. Assume $x_j \neq 0$ all j, or else drop this term from both sums. Let $w_j = \frac{|x_j|^p}{\|x\|_p^p}$ so that $\sum_j w_j = 1$ and $w_j \geqslant 0$. Since the function $x \mapsto x^q$ is convex,

$$\left(\sum_j \frac{|y_j|}{|x_j|^{p-1}} w_j\right)^{\frac{p-1}{p}} \leqslant \sum_j \left(\frac{|y_j|}{|x_j|^{p-1}}\right)^{\frac{p}{p-1}} w_j.$$

Unwrapping this inequality proves that

$$\sum |x_j y_j| \leqslant \|x\|_p \|y\|_q.$$

$\square$

Problem 1. Find

$$\min_{a,b\in\mathbb{R}} \max(a^2+b, b^2+a).$$

Problem 2. Prove that for all real numbers x,

$$2^x + 3^x - 4^x + 6^x - 9^x \leqslant 1.$$

Problem 3. Find all triples (x, y, z) of real numbers which are simultaneous solutions of the system

$$\frac{4x^2}{4x^2+1} = y, \ \frac{4y^2}{4y^2+1} = z, \ \frac{4z^2}{4z^2+1} = x.$$

Problem 4. Let n be an even positive integer. Prove that for any real number x there are at least $2^{\frac{n}{2}}$ choices of the signs $+$ and $-$ such that

$$\pm x^n \pm x^{n-1} \pm \cdots \pm x < \frac{1}{2}.$$

Problem 5. If $a_1 + a_2 + \cdots + a_n = n$ prove that $a_1^4 + \cdots + a_n^4 \geqslant n$.

Problem 6. Let $a_1, a_2, ..., a_n$ be distinct real numbers. Find the maximum of

$$a_1 a_{\sigma(1)} + \cdots + a_n a_{\sigma(n)}$$

over all permutations σ of $\{1, 2, ..., n\}$.

Problem 7. Prove that the finite sequence $a_0, a_1, ..., a_n$ of positive real numbers is a geometric progession if and only if

$$(a_0 a_1 + a_1 a_2 + \cdots + a_{n-1} a_n)^2 = (a_0^2 + \cdots + a_{n-1}^2)(a_1^2 + \cdots + a_n^2).$$

Problem 8. Let $P(z)$ be a polynomial with real coefficients whose roots can be covered by a disk of radius R. Prove that for any real number k, the roots of the polynomial $nP(z) - kP'(z)$ can be covered by a disk of radius $R + |k|$, where n is the degree of $P(z)$ and $P'(z)$ is the derivative.

Problem 9. Let $V_1, ..., V_m$ and $W_1, ..., W_m$ be isometries of $\mathbb{R}^n$ (m, n positive integers). Assume that for all x with $\|x\| \leqslant 1$, $\|V_i x - W_i x\| \leqslant 1$, $i = 1, 2, ..., n$. Prove that

$$\left\| \left(\prod_{i=1}^m V_i \right) x - \left(\prod_{i=1}^m W_i \right) x \right\| \leqslant m,$$

for all x with $\|x\| \leqslant 1$.

Problem 10. Which number is larger,

$$\prod_{n=1}^{25} \left(1 - \frac{n}{365}\right) \quad \text{or} \quad \frac{1}{2}?$$

Problem 11. Let $a_1, a_2, ..., a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n < 1$. Prove that

$$\frac{a_1 a_2 \cdots a_n (1 - (a_1 + \cdots + a_n))}{(a_1 + \cdots + a_n)(1 - a_1) \cdots (1 - a_n)} \leqslant \frac{1}{n^{n+1}}.$$

Problem 12. Let a, b, c be nonnegative real numbers such that $a + b + c = 1$. Prove that

$$4(ab + bc + ac) - 9abc \leqslant 1.$$

Problem 13. Let $x_1, x_2, ..., x_n$, $n \geqslant 2$, be positive numbers such that $x_1 + x_2 + \cdots + x_n = 1$. Prove that

$$\left(1 + \frac{1}{x_1}\right)\left(1 + \frac{1}{x_2}\right) \cdots \left(1 + \frac{1}{x_n}\right) \geqslant (n+1)^n.$$

Problem 14. Let $x_1, x_2, ..., x_n$ be n real numbers such that $0 < x_j \leqslant \frac{1}{2}$, for $1 \leqslant j \leqslant n$. Prove the inequality

$$\frac{\prod_{j=1}^n x_j}{\left(\sum_{j=1}^n x_j\right)^n} \leqslant \frac{\prod_{j=1}^n (1 - x_j)}{\left(\sum_{j=1}^n (1 - x_j)\right)^n}.$$

Problem 15. What is the maximal value of the expression $\sum_{i<j} x_i x_j$ if $x_1, x_2, ..., x_n$ are non-negative integers whose sum is equal to m?

Problem 16. Prove for each $n \geqslant 1$,

$$\left(1 + \frac{1}{n}\right)^n \leqslant \left(1 + \frac{1}{n+1}\right)^{n+1}.$$

Problem 17. Show that $\sin^2 x < \sin x^2$ for $0 < x < \sqrt{\frac{\pi}{2}}$.

Problem 18. Prove that among all convex n-gons inscribed in a circle, the regular n-gon maximizes the area.

Problem 19. Show that if $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx \leqslant 2 \left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |f'(x)|^2 dx\right)^{\frac{1}{2}}.$$

Problem 20. Suppose that $-\infty < \alpha \leqslant A < \infty$ and $-\infty < \beta \leqslant B < \infty$ and suppose that functions f and g satisfy the bounds

$$\alpha \leqslant f(x) \leqslant A \quad \text{and} \quad \beta \leqslant g(x) \leqslant B$$

for all $0 \leqslant x \leqslant 1$. Show that one has the bound

$$\left|\int_0^1 f(x)g(x)dx - \int_0^1 f(x)dx \int_0^1 g(x)dx\right| \leqslant \frac{1}{4}(A - \alpha)(B - \beta).$$

Problem 21.

a. Suppose that $-1 \leqslant x_1 < x_2 < \cdots < x_n \leqslant 1$, and show that

$$\sum_{1 \leqslant j < k \leqslant n} \frac{1}{x_k - x_j} \geqslant \frac{1}{8} n^2 \log n.$$

b. Show that for any permutation $\sigma : [n] \to [n]$ one has the bound

$$\max_{1 < k \leqslant n} \sum_{j=1}^{k-1} \frac{1}{|x_{\sigma(k)} - x_{\sigma(j)}|} \geqslant \frac{1}{8} n \log n.$$