SEQUENCES AND SERIES

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Let $F_0 = 0, F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ be the Fibonacci sequence. Let

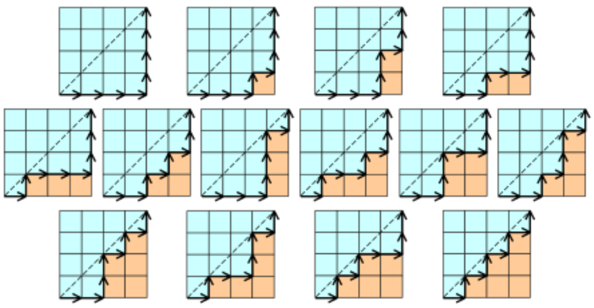
$$f(x) = \sum_{n=0}^{\infty} F_n x^n,$$

which trivially converges for $|x| < \frac{1}{2}$ (why?). Write $f(x) = x + \sum_{n=2}^{\infty}(F_{n-1} + F_{n-2})x^n = x + xf(x) + x^2 f(x)$, so, for $|x| < \frac{1}{2}$, $\phi = \frac{1+\sqrt{5}}{2}$

$$f(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-\phi x)(1+\phi^{-1}x)} = \frac{A}{1-\phi x} + \frac{B}{1+\phi^{-1}x}.$$

Plugging in $x = \phi^{-1}$ and $x = -\phi$ obtains $A = \frac{\phi^{-1}}{1+\phi^{-2}} = \frac{1}{\sqrt{5}}$, $B = -\frac{\phi}{1+\phi^2} = -\frac{1}{\sqrt{5}}$. Expanding the geometric series, then equating x^n coefficients, $F_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi)^{-n})$.

The Dyck paths of length $2n$ are lattice paths from $(0,0)$ to (n,n) which move only right and up and are contained on or below the diagonal. Here's a picture from Wikipedia:



The number of length $2n$ Dyck paths is the Catalan number C_n.

Theorem 1. *The Catalan numbers C_n satisfy $C_0 = 1$ and, for $n \geqslant 0$, $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$, and the formula $C_n = \frac{\binom{2n}{n}}{n+1}$.*

Proof. The first move in a Dyck path must move right. Suppose after $2j+2$ moves it first touches the diagonal. Then the moves between 1 and $2j+1$ constitute a Dyck path of length $2j$ with the diagonal shifted by one unit right. The remaining moves are a Dyck path of length $2(n-j)$. The number of choices for each Dyck path is C_j, C_{n-j} respectively, hence the recurrence formula.

To prove the closed formula, define $c(x) = \sum_{n=0}^{\infty} C_n x^n$. Since $C_n \leqslant 2^{2n}$ (why?) this converges for $|x| < \frac{1}{4}$. It satisfies $c(x) = 1 + xc(x)^2$, or $c(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$. Since $c(0) = 1$, $c(x) = \frac{1-\sqrt{1-4x}}{2x}$. Expand $(1-4x)^{\frac{1}{2}}$ in Taylor series about 0 to obtain[1]

$$(1-4x)^{\frac{1}{2}} = \sum_{j=0}^{\infty}(-4x)^j \binom{\frac{1}{2}}{j} = \sum_{j=0}^{\infty} \frac{(-4x)^j}{j!} \left(\frac{1}{2}\right)^{\underline{j}} = 1 - \sum_{j=1}^{\infty} \frac{(2x)^j}{j!} \frac{(2j-2)!}{2^{j-1}(j-1)!}$$

Hence, $\frac{1-(1-4x)^{\frac{1}{2}}}{2x} = \sum_{j=0}^{\infty} \frac{(2j)!}{j!(j+1)!} x^j$, so $C_n = \frac{\binom{2n}{n}}{n+1}$.

□

[1] $x^{\underline{j}} = x(x-1)\cdots(x-j+1)$

Problem 1. Let $(x_n)_{n\geqslant 1}$ be a sequence of real numbers satisfying

$$x_{n+m} \leqslant x_n + x_m, \qquad n, m \geqslant 1.$$

Show that $\lim_{n\to\infty} \frac{x_n}{n} = \inf_{n\geqslant 1} \frac{x_n}{n}$.

Problem 2. Prove that

$$\lim_{n\to\infty} n^2 \int_0^{\frac{1}{n}} x^{x+1} dx = \frac{1}{2}.$$

Problem 3. Let $p(x) = x^2 - 3x + 2$. Show that for any positive integer n there exist unique numbers a_n and b_n such that the polynomial $q_n(x) = x^n - a_n x - b_n$ is divisible by $p(x)$.

Problem 4. Let $(x_n)_{n\geqslant 0}$ be defined by the recurrence relation $x_{n+1} = ax_n + bx_{n-1}$, with $x_0 = 0$. Show that the expression $x_n^2 - x_{n-1}x_{n+1}$ depends only on b and x_1, but not on a.

Problem 5. Prove convergence of the sequence $(a_n)_{n\geqslant 1}$ defined by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1), \qquad n \geqslant 1.$$

Problem 6. Prove convergence of the sequence

$$a_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}}}, \qquad n \geqslant 1.$$

Problem 7. Let $0 < a < b$ be two real numbers. Define the sequences $(a_n)_n$ and $(b_n)_n$ by $a_0 = a, b_0 = b$, and

$$a_{n+1} = \sqrt{a_n b_n}, \qquad b_{n+1} = \frac{a_n + b_n}{2}, \qquad n \geqslant 0.$$

Prove that the sequences $\{a_n\}$ and $\{b_n\}$ are convergent and have the same limit.

Problem 8. Prove that for $n \geqslant 2$, the equation $x^n + x - 1 = 0$ has a unique root in the interval $[0, 1]$. If x_n denotes this root, prove that the sequence $(x_n)_{n\geqslant 1}$ has a limit and determine the limit.

Problem 9. Let $f : [a, b] \to [a, b]$ be an increasing function. Show that there exists $\xi \in [a, b]$ such that $f(\xi) = \xi$.

Problem 10. Let $S = \{x_1, x_2, ..., x_n, ...\}$ be the set of all positive integers that do not contain the digit 9 in their decimal representation. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{x_n} < 80.$$

Problem 11. For a nonnegative integer k, define $S_k(n) = 1^k + 2^k + \cdots + n^k$. Prove that

$$1 + \sum_{k=0}^{r-1} \binom{r}{k} S_k(n) = (n+1)^r.$$

Problem 12. Compute the product

$$\left(1 - \frac{4}{1}\right)\left(1 - \frac{4}{9}\right)\left(1 - \frac{4}{25}\right)\cdots.$$

Problem 13. Let x be a positive number less than 1. Compute the product

$$\prod_{n=0}^{\infty} \left(1 + x^{2^n}\right).$$