REAL ANALYSIS

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Problem 1. Let $n > 1$ be an integer, and let $f : [a, b] \to \mathbb{R}$ be a continuous function, n-times differentiable on (a, b), with the property that the graph of f has $n + 1$ collinear points. Prove that there exists a point $c \in (a, b)$ with the property that $f^{(n)}(c) = 0$.

Problem 2. Let α be a real number such that n^α is an integer for every positive integer n. Prove that α is a nonnegative integer.

Problem 3. Let $x_1, x_2, ..., x_n$ be real numbers. Find the real number a that minimizes the expression

$$|a - x_1| + |a - x_2| + \cdots + |a - x_n|.$$

Problem 4. Let a_i, $i = 1, 2, ..., n$ be non-negative numbers with $\sum_{i=1}^n a_i = 1$, and let $0 < x_i \leqslant 1$, $i = 1, 2, ..., n$. Prove that

$$\sum_{i=1}^n \frac{a_i}{1 + x_i} \leqslant \frac{1}{1 + x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}.$$

Problem 5. Compute

$$\lim_{n\to\infty} \left(\frac{2^{1/n}}{n+1} + \frac{2^{2/n}}{n + \frac{1}{2}} + \cdots + \frac{2^{n/n}}{n + \frac{1}{n}} \right).$$

Problem 6. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = 1.$$

Prove that

$$\int_0^1 f^2(x)dx \geqslant 4.$$

Problem 7. Let $a_1, a_2, ..., a_n$ be positive real numbers and let $x_1, x_2, ..., x_n$ be real numbers such that $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$. Prove that

$$\sum_{i,j} x_i x_j |a_i - a_j| \leqslant 0.$$

Prove that equality holds if and only if there exits a partition of the set $\{1, 2, ..., n\}$ into the disjoint sets $A_1, ..., A_k$ such that if i and j are in the same set, then $a_i = a_j$ and also $\sum_{j\in A_i} x_j = 0$ for $i = 1, 2, ..., k$.

Problem 8. Let $f : [0, \infty) \to [0, \infty)$ be a continuous, strictly increasing function with $f(0) = 0$. Prove that

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geqslant ab$$

for all positive numbers a and b, with equality if and only if $b = f(a)$. Here f^{-1} denotes the inverse of the function f.

Problem 9. Prove that any continuously differentiable function $f : [a, b] \to \mathbb{R}$ for which $f(a) = 0$ satisfies the inequality

$$\int_a^b f(x)^2 dx \leqslant (b-a)^2 \int_a^b f'(x)^2 dx.$$

Problem 10. For $a > 0$, prove that

$$\int_{-\infty}^{\infty} e^{-x^2} \cos ax dx = \sqrt{\pi} e^{-a^2/4}.$$

Problem 11. Prove that for every $0 < x < 2\pi$ the following formula is valid:

$$\frac{\pi - x}{2} = \frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots .$$

Hence conclude

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}, \qquad x \in (0, \pi).$$

Problem 12. Let $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n = m$ be positive integers. Denote by b_k the number of those a_i for which $a_i \geqslant k$. Prove that

$$a_1 + a_2 + \cdots + a_n = b_1 + \cdots + b_m.$$