PROBABILITY

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In the *probabilistic method* one proves that an object exists with some characteristic by assigning a probability to all objects and verifying that the characteristic has positive probability.

The *Ramsey number* $R(k, \ell)$ is the smallest number n such that in any two-coloring red/blue of the edges of the complete graph on n vertices there is either a red complete subgraph on k vertices or a blue complete subgraph on ℓ vertices.

Theorem 1. *If* $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ *then* $R(k,k) > n$.

Proof. Given a complete graph on n vertices, color the edges red or blue independently with equal probability. Given any set of k vertices R, the probability that all edges connecting the vertices are the same color is $2^{1-\binom{k}{2}}$. The number of such sets of k vertices is $\binom{n}{k}$. Hence

$$\begin{aligned}\textbf{Prob}\,(\text{no monochromatic } K_k) &\geqslant 1 - \sum_R \textbf{Prob}\,(R \text{ monochromatic}) \\ &= 1 - \binom{n}{k}2^{1-\binom{k}{2}}.\end{aligned}$$

Hence if $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ then there is a coloring of K_n with two colors with no monochromatic K_k subgraph. □

Problem 1. Find the probability that in a group of n people there are two with the same birthday, ignoring leap years.

Problem 2. What is the probability that a permutation of the first n positive integers has the numbers 1 and 2 within the same cycle.

Problem 3. An unbiased coin is tossed n times. Find a formula, in closed form, for the expected value of $|H - T|$, where H is the number of heads, and T is the number of tails.

Problem 4. Find the probability that in the process of repeatedly flipping a coin, one will encounter a run of 5 heads before one encounters a run of 2 tails.

Problem 5. Given the independent events $A_1, A_2, ..., A_n$ with probabilities $p_1, p_2, ..., p_n$, find the probability that an odd number of these events occur.

Problem 6. A coin is tossed n times. What is the probability that two heads will turn up in succession somewhere in the sequence.

Problem 7. What is the probability that the sum of two randomly chosen numbers in the interval $[0, 1]$ does not exceed 1 and their product does not exceed $\frac{2}{9}$?

Problem 8. Let α and β be given positive real numbers, with $\alpha < \beta$. If two points are selected at random from a straight line segment of length β, what is the probability that the distance between them is at least α?

Problem 9. What is the probability that three points selected at random on a circle lie on a semicircle?

Problem 10. If a needle of length 1 is dropped at random on a surface ruled with parallel lines at distance 2 apart, what is the probability that the needle will cross one of the lines?

Problem 11. Four points are chosen uniformly at random in the interior of a circle. Find the probability that they are the vertices of a convex quadrilateral.

Problem 12. Let $A_1, ..., A_n \subset \{1, ..., m\}$ with $\sum_{i=1}^n 2^{-|A_i|} < 1$. Paul and Carole play a game in which they alternately choose elements of $\{1, 2, ..., m\}$, with Paul choosing first. Carole wins if she holds a complete set A_i at the end, otherwise Paul wins. Give a winning strategy for Paul.

Problem 13. Let $v_1, v_2, ..., v_n \in \mathbb{R}^n$ with $|v_i| = 1$. Then there are $\epsilon_1, ..., \epsilon_n \in \{-1, 1\}$ such that
$$|\epsilon_1 v_1 + \cdots + \epsilon_n v_n| \leqslant \sqrt{n},$$
and also there exist $\epsilon_1, ..., \epsilon_n \in \{-1, 1\}$ such that
$$|\epsilon_1 v_1 + \cdots + \epsilon_n v_n| \geqslant \sqrt{n}.$$

Problem 14. Let $a_1, a_2, ..., a_n > 0$ and define the power series
$$\frac{1}{(1 - a_1 x)(1 - a_2 x)...(1 - a_n x)} = 1 + nq_1 x + \binom{n+1}{2} q_2 x^2 + ... + \binom{n+r-1}{r} q_r x^r + ...$$
Prove that for $r \geqslant 1$, $q_r^2 < q_{r-1} q_{r+1}$ unless all the a_j are equal.

Problem 15. Show that for complex numbers $z_1, ..., z_n$,
$$\frac{1}{\pi} \sum_{i=1}^n |z_i| \leqslant \max_{I \subset \{1,2,...,n\}} \left| \sum_{i \in I} z_i \right|.$$

Problem 16. (Selberg's inequality) Let $u, v_1, ..., v_n$ be non-zero elements of a real or complex inner product space. Prove that
$$\|u\|^2 \geqslant \sum_{j=1}^n \frac{|\langle u, v_j \rangle|^2}{\sum_{k=1}^n |\langle v_j, v_k \rangle|}.$$