POLYNOMIALS

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A polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ factors into linear factors over $\mathbb{C}$, $P(x) = a_n(x - x_1)...(x - x_n)$. The coefficients are elementary symmetric functions in the roots:

$$-\frac{a_{n-1}}{a_n} = x_1 + \cdots + x_n$$
$$\frac{a_{n-2}}{a_n} = x_1x_2 + \cdots + x_{n-1}x_n$$
$$\vdots$$
$$(-1)^n\frac{a_0}{a_n} = x_1x_2\cdots x_n.$$

Theorem 1 (Eisenstein's criterion)**.** *Given a polynomial $P(x) = a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$ with integer coefficients, suppose that there exists a prime number p such that a_n is not divisible by p, a_k is divisible by p for $k = 0, 1, 2, ..., n-1$ and a_0 is not divisible by p^2. Then $P(x)$ is irreducible over $\mathbb{Z}[x]$.*

Proof. By contradiction. Suppose $P(x) = Q(x)R(x)$ and both have degree at least 1. Let

$$Q(x) = b_kx^k + b_{k-1}x^{k-1} + \cdots + b_0,$$
$$R(x) = c_{n-k}x^{n-k} + c_{n-k-1}x^{n-k-1} + \cdots + c_0.$$

Multiplying the polynomials obtains the identities

$$a_i = \sum_{j=0}^{i} b_jc_{i-j}, \qquad i = 0, 1, 2, ...$$

Since $b_0c_0 = a_0$ is divisible by p but not p^2, exactly one of b_0, c_0 is divisible by p. Say $p|b_0$. We show that $p|b_i$ for all i by induction. If $p|b_0, b_1, ..., b_{i-1}$ then

$$a_i = b_0c_i + b_1c_{i-1} + \cdots + b_ic_0.$$

All but the b_ic_0 term is divisible by p, so b_i is also. Since $p|Q(x)$, $p|P(x)$, a contradiction. □

Problem 1. Find a polynomial with integer coefficients that has the zero $2^{\frac{1}{2}} + 3^{\frac{1}{3}}$.

Problem 2. Let a, b, c be real numbers. Show that $a \geqslant 0$, $b \geqslant 0$ and $c \geqslant 0$ if and only if $a + b + c \geqslant 0$, $ab + bc + ca \geqslant 0$, and $abc \geqslant 0$.

Problem 3. Determine all polynomials $P(x)$ with real coefficients satisfying $(P(x))^n = P(x^n)$ for all $x \in \mathbb{R}$, where $n > 1$ is a fixed integer.

Problem 4. Let $a \in \mathbb{C}$ and $n \geqslant 2$. Prove that the polynomial equation $ax^n + x + 1 = 0$ has a root of absolute value less than or equal to 2.

Problem 5. Let $P(z)$ be a polynomial of even degree n, all of whose zeros have absolute value 1 in the complex plane. Set $g(z) = \frac{P(z)}{z^{\frac{n}{2}}}$. Show that all roots of the equation $g'(z) = 0$ have absolute value 1.

Problem 6. Prove that the polynomial

$$P(x) = x^{101} + 101x^{100} + 102$$

is irreducible over $\mathbb{Z}[x]$.

Problem 7. Prove that for every prime number p, the polynomial

$$P(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is irreducible over $\mathbb{Z}[x]$.

Problem 8. Prove that for every positive integer n, the polynomial $P(x) = x^{2^n} + 1$ is irreducible over $\mathbb{Z}[x]$.

Problem 9. Prove that for any distinct integers $a_1, a_2, ..., a_n$ the polynomial

$$P(x) = (x - a_1)(x - a_2) \cdots (x - a_n) - 1$$

cannot be written as a product of two nonconstant polynomials with integer coefficients.

Problem 10. Prove that for every positive integer n,

$$\tan \frac{\pi}{2n+1} \tan \frac{2\pi}{2n+1} \cdots \tan \frac{n\pi}{2n+1} = \sqrt{2n+1}.$$

Problem 11. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with complex coefficients, with $a_0 \neq 0$, and with the property that there exists an m such that

$$\left| \frac{a_m}{a_0} \right| \geqslant \binom{n}{m}.$$

Prove that $P(x)$ has a zero of absolute value less than 1.

Problem 12. Compute the $n \times n$ determinant

$$\det \begin{pmatrix} x & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2x & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2x & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 2x \end{pmatrix}.$$

Problem 13. Given the polynomial $P(x, y, z)$ prove that the polynomial

$$Q(x, y, z) = P(x, y, z) + P(y, z, x) + P(z, x, y) - P(x, z, y) - P(y, x, z) - P(z, y, x)$$

is divisible by $(x - y)(y - z)(z - x)$.

Problem 14. Solve the system

$$x + y + z = 1,$$
$$xyz = 1,$$

given that x, y, z are complex numbers of absolute value 1.