

## INVARIANTS

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An *invariant* of a system is a quantity which remains constant as the system evolves. Also useful in studying systems are quantities which only increase or decrease as the system evolves.

A doubly stochastic  $n \times n$  matrix has non-negative entries with rows and columns that each sum to 1. A permutation matrix is an  $n \times n$  0-1 matrix with one 1 in each row and column.

**Theorem 1** (Birkhoff's Theorem). *Every doubly stochastic  $n \times n$  matrix is a convex combination of permutation matrices.*

*Proof.* The property of being doubly stochastic is preserved under convex combinations, since if  $M_1$  and  $M_2$  are doubly stochastic, for  $0 \leq c \leq 1$ ,  $cM_1 + (1 - c)M_2$  still has row and column sums equal to 1.

We reduce to the following claim: given any doubly stochastic matrix  $M$  there is a permutation matrix  $P$  and a  $\delta > 0$  such that  $M - \delta P$  has non-negative entries.

We first check that the claim proves the theorem. The sum of the rows and columns of  $M - \delta P$  are each  $1 - \delta$ . Choose  $\delta$  maximal so that the entries of  $M - \delta P$  are non-negative. If  $\delta = 1$  then  $M$  is a permutation matrix. Otherwise,  $M' = \frac{1}{1-\delta}(M - \delta P)$  is again doubly stochastic and has fewer non-zero entries than  $M$  had. Since a doubly stochastic matrix with  $n$  non-zero entries is a permutation matrix, repeating this process at most  $n^2 - n$  times guarantees we reach a permutation matrix.

To prove the claim, given the matrix  $M$ , in each non-zero entry  $m_{ij}$  assign a variable  $x_{ij}$  and let  $X$  be the matrix with zero entries where  $M$  is 0, and variable  $x_{ij}$  where  $M$  is non-zero. We claim  $\det(X) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n x_{i\sigma(i)} \neq 0$ . Note that this guarantees that  $M > \delta P$  for some permutation matrix  $P$  and some  $\delta > 0$ . To check the determinant, suppose for a contradiction that  $\det(X) = 0$  so that some collection of rows are linearly dependent, say w.l.o.g.  $r_1, \dots, r_k$  are a minimal set of dependent rows, that is  $a_1 r_1 + \dots + a_k r_k = 0$  with  $a_1, \dots, a_k$  non-zero and no smaller set is dependent. Since the non-zero entries are variables, this can only occur if the number of columns which are non-zero in  $r_1, \dots, r_k$  is less than  $k$ . To see this, verify that  $r_1, \dots, r_{k-1}$  are l.i. by checking columns  $c_1, \dots, c_{k-1}$ . Then  $r_k$  has its entries contained in  $c_1, \dots, c_{k-1}$  since the remaining variables outside these columns may be set to 0 in  $r_1, \dots, r_{k-1}$ . Then  $r_1, \dots, r_{k-1}$  have their entries 0 outside  $c_1, \dots, c_{k-1}$ , since if one did not, one of its entries outside could be set to 0 while all others are not, so that it does not form part of the linear combination in  $r_k$ , contradicting minimality. Summing in columns first, the sum of the entries in  $r_1, \dots, r_k$  is at most  $k - 1$ . But each row sums to 1, contradiction.  $\square$

*Problem 1.* Place a knight on each square of an  $7 \times 7$  chessboard. Is it possible that each knight can simultaneously make a legal move?

*Problem 2.* Let  $n$  be an odd integer greater than 1, and let  $A$  be an  $n$ -by- $n$  symmetric matrix such that each row and each column of  $A$  consists of some permutation of the integers  $1, \dots, n$ . Show that each one of the integers  $1, \dots, n$  must appear in the main diagonal of  $A$ .

*Problem 3.* The entries of a matrix are real numbers of absolute value less than or equal to 1, and the sum of the elements in each column is 0. Prove that we can permute the elements of each column in such a way that the sum of the elements in each row will have absolute value less than or equal to 2.

*Problem 4.* An ordered triple of numbers is given. It is permitted to perform the following operation on the triple: to change two of them, say  $a$  and  $b$ , to  $(a+b)/\sqrt{2}$  and  $(a-b)/\sqrt{2}$ . Is it possible to obtain the triple  $(1, \sqrt{2}, 1 + \sqrt{2})$  from the triple  $(2, \sqrt{2}, 1/\sqrt{2})$  using this operation?

*Problem 5.* There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than one stone, throw away one stone from the heap, then divide it into two smaller (not necessarily equal) heaps. Is it possible to reach a situation in which all the heaps on the table contain exactly 3 stones by performing the operation finitely many times.

*Problem 6.* On an arbitrarily large chessboard consider a generalized knight that can jump  $p$  squares in one direction and  $q$  in the other,  $p, q > 0$ . Show that such a knight can return to its initial position only an *even* number of jumps.

*Problem 7.* In the squares of a  $3 \times 3$  chessboard are written the signs  $+$  and  $-$  as described below. Consider the operations in which one is allowed to simultaneously change all signs in some row or column. Can one change the given configuration to the other one by applying such operations finitely many times?

+	+	-	-	-	+
+	+	-	+	-	-
-	-	+	-	-	+

*Problem 8.* A real number is written in each square of an  $n \times n$  chessboard. We can perform the operation of changing all signs of the numbers in a row or a column. Prove that by performing this operation a finite number of times we can produce a new table for which the sum of each row or column is non-negative.

*Problem 9.* Several positive integers are written on a blackboard. One can erase any two distinct integers and write their greatest common divisor and least common multiple on the board instead. Prove that eventually the numbers will stop changing.

*Problem 10.* Consider the integer lattice in the plane, with one pebble at the origin. We play a game in which at each step one pebble is removed from a node of the lattice and two new pebbles are placed at two neighboring nodes, provided that those nodes are unoccupied. Prove that at any time there will be a pebble at distance at most 5 from the origin.