INDUCTION AND PIGEONHOLE

ROBERT HOUGH

To prove a statement $P(n)$ for all positive integers n by induction, prove that $P(1)$ is true, and prove that $P(n)$ implies $P(n+1)$. Hence the statement $1+2+...+n = \frac{n(n+1)}{2}$ can be checked:

$$\text{Base case:}1 = \frac{1(1+1)}{2},$$
$$\text{Inductive step:}1+2+...+n = \frac{n(n+1)}{2}$$
$$\Rightarrow 1+2+...+n+(n+1) = \frac{n(n+1)}{2}+n+1 = \frac{(n+1)(n+2)}{2}.$$

The pigeonhole principle states that a function $f : S \to T$ from a finite set S to a finite set T with $|S| > |T|$ has some $t \in T$ with $|f^{-1}(t)| \geqslant 2$. For instance, let S be a set of size 10 contained in $\{1, 2, ..., 20\}$. The number of pairs $s_1 < s_2$ from S is $\binom{10}{2} = 45$. Meanwhile, all sums are between 3 and 39. Hence there are two pairs with the same sum.

Example 1. Prove $2!4!...(2n)! \geqslant ((n+1)!)^n$.

Proof. For $n = 1$ this reduces to $2! \geqslant (2!)^1$. Suppose that it holds for $n \geqslant 1$. Then, by the inductive assumption,

$$2!4!...(2n+2)! \geqslant ((n+1)!)^n(2n+2)!.$$

Since $\frac{((n+2)!)^{n+1}}{((n+1)!)^n} = (n+1)!(n+2)^{n+1} \leqslant (n+1)!(n+2)(n+3)...(2n+2) = (2n+2)!$ it follows that

$$2!4!...(2n+2)! \geqslant ((n+2)!)^{n+1},$$

completing the inductive step. □

Example 2. Prove that no seven positive integers, not exceeding 24, can have all different subsets have different sums.

Proof. There are $2^7 - 1 = 127$ different non-empty subsets. If we can show that these subsets have sums in an interval of length less than 127 then the conclusion will follow from the pigeonhole principle.

If all subsets have a different sum, then each of the integers is distinct. Let $1 \leqslant m_1 < m_2 < m_3 < m_4 < m_5 < m_6 < m_7 \leqslant 24$ be the integers. All of the non-empty subset sums are in the range $[m_1, m_1 + \cdots + m_7]$ which has length $\leqslant 1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7$. Note that the six integers $m_2, ..., m_7$ have $\binom{6}{2} = 15$ different pairs, each of which must have a distinct sum. In particular, by the pigeonhole principle, $m_2 + m_3 \leqslant m_6 + m_7 - 14 \leqslant 23 + 24 - 14 = 33$. Also, $m_4 + m_5 \leqslant 21 + 22 = 43$. Thus $m_2 + ... + m_7 \leqslant 33 + 43 + 47 = 123$ which proves the claim. □

Problem 1. Let X be a real number. Prove that among the set $X, 2X, ..., (n-1)X$ there is a number which differs from an integer by at most $\frac{1}{n}$.

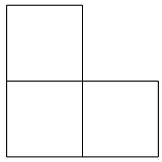
Problem 2. Prove that for any $x_1, ..., x_n$, $n \geqslant 1$,

$$|\sin x_1| + \cdots + |\sin x_n| + |\cos(x_1 + \cdots + x_n)| \geqslant 1.$$

Problem 3. Let n be a positive integer. Prove that

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{3}{2}.$$

Problem 4. Prove that for any $n \geqslant 1$, a $2^n \times 2^n$ checkerboard with 1×1 corner square removed can be tiled by pieces of the form below.



Problem 5. Show that every positive integer can be written as the sum of distinct terms of the Fibonacci sequence.

Problem 6. Prove that every positive integer can be represented as $\pm 1^2 \pm 2^2 \pm \cdots \pm n^2$ for some positive integer n and some choice of signs.

Problem 7. Given 51 distinct positive integers strictly less than 100, prove that some two of them sum to 99.

Problem 8. Let $x_1, x_2, x_3, ...$ be a sequence of integers such that

$$1 = x_1 < x_2 < x_3 < \cdots, \qquad x_{n+1} \leqslant 2n$$

for $n = 1, 2, 3, ...$ Show that every positive integer k is equal to $x_i - x_j$ for some i and j.

Problem 9. Prove that there is a positive term of the Fibonacci sequence divisible by 1000.

Problem 10. Inside a circle of radius 4 are chosen 61 points. Show that among them there are two at distance at most $\sqrt{2}$ from each other.

Problem 11. Inside the unit square lie several circles the sum of whose circumferences is equal to 10. Prove that there exist infinitely many lines each of which intersects at least four of the circles.