GEOMETRY AND COMPLEX NUMBERS

ROBERT HOUGH

Recall Euler's formula $e^{ix} = \cos x + i \sin x$. This is a complex number of modulus 1. This formula is useful for proving trigonometric identities, for instance

$$\cos 2x + i \sin 2x = e^{i2x} = (e^{ix})^2 = (\cos x + i \sin x)^2 = \cos^2 x - \sin^2 x + 2i \cos x \sin x.$$

Matching up the real and imaginary parts gives the double angle formulas. Can you calculate the numerical value of the series

$$1 - \frac{1}{4} + \frac{1}{6} - \frac{1}{9} + \frac{1}{11} - \frac{1}{14} + ... = \sum_{k=0}^{\infty} \frac{1}{5k+1} - \frac{1}{5k+4}?$$

Multiplying a complex number z by e^{ix} rotates z by an amount x radians counter-clockwise about the origin. Also, complex numbers add like vectors. This makes complex numbers useful in geometry.

The numbers $e^{\frac{2\pi ij}{n}}$ with $0 \leqslant j < n$ are the nth roots of unity, and the vertices of a regular n-gon. Recall the factorization

$$1 + x + ... + x^{n-1} = \prod_{j=1}^{n-1} \left(x - e^{\frac{2\pi ij}{n}} \right).$$

Recall that the coefficients of a polynomial are (up to sign) elementary symmetric polynomials in the roots. Thus this formula gives elementary symmetric polynomials in the nth roots of unity other than 1.

The Fundamental Theorem of Algebra states that every degree n polynomial has n complex roots, counted with multiplicity. Check that the roots of the derivative lie within the complex hull of the roots of the polynomial.

Problem 1. Consider a unit vector starting at the origin and pointing in the direction of the tangent vector to a continuously differentiable curve in 3 dimensional space. The endpoint of the vector describes the spherical image of the curve (on the unit sphere). Show that if the curve is closed, then its spherical image intersects every great circle of the unit sphere.

Problem 2. Two convex polygons are placed one inside the other. Prove that the perimeter of the interior polygon that lies inside is smaller.

Problem 3. Centered at every point with integer coordinates in the plane there is a disk with radius $\frac{1}{1000}$. Prove that there is an equilateral triangle whose vertices lie inside different disks.

Problem 4. A rectangle R is tiled by finitely many rectangles, each of which has at least one side of integer length. Prove that R has at least one side of integer length.

Problem 5. Someone has drawn two squares of radius .9 inside a circle of radius 1. Prove that the two squares overlap.

Problem 6. Let $A_1, A_2, ..., A_n$ be distinct points in the plane, and let m be the number of midpoints of all segments they determine. What is the smallest value that m can have?

Problem 7. Let $A_1A_2...A_n$ be a regular polygon with circumradius equal to 1. Find the maximum value of $\prod_{k=1}^n PA_k$ as P ranges over the circumcircle.

Problem 8. Let $A_0, A_1, ..., A_n$ be the vertices of a regular n-gon inscribed in the unit circle. Prove that

$$A_0A_1 \cdot A_0A_2 \cdots A_0A_{n-1} = n.$$

Problem 9. Find all regular polygons that can be inscribed in an ellipse with unequal semiaxes.

Problem 10. Find the maximum number of points on a sphere of radius 1 in $\mathbb{R}^n$ such that the distance between any two points is strictly greater than $\sqrt{2}$.

Problem 11. Prove the identity

$$\left(\frac{1 + i\tan t}{1 - i\tan t}\right)^n = \frac{1 + i\tan nt}{1 - i\tan nt}.$$

Problem 12. Prove the identity

$$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots = 2^{n/2}\cos\frac{n\pi}{4}.$$

Problem 13. Compute the sum

$$\binom{n}{1}\cos x + \binom{n}{2}\cos 2x + \cdots + \binom{n}{n}\cos nx.$$

Problem 14. Prove that

$$\frac{1}{\sin 45 \sin 46} + \frac{1}{\sin 47 \sin 48} + \cdots + \frac{1}{\sin 133 \sin 134} = \frac{1}{\sin 1}.$$

Problem 15. Prove that

$$\left(1 - \frac{\cos 61}{\cos 1}\right)\left(1 - \frac{\cos 62}{\cos 2}\right)\cdots\left(1 - \frac{\cos 119}{\cos 59}\right) = 1.$$