FUNCTIONAL EQUATIONS

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Cauchy's functional equation is

$$f(x+y) = f(x) + f(y),$$

where $f : \mathbb{R} \to \mathbb{R}$. Substitute $y = 0$ to obtain $f(x) = f(x) + f(0)$ so $f(0) = 0$. Iterating the equation, for integers n, $f(nx) = nf(x)$, and hence $f\left(\frac{m}{n}x\right) = \frac{m}{n}f(x)$. Hence the value of f at rationals is determined by $f(1)$, $f(q) = qf(1)$. If f is continuous, then $f(x) = xf(1)$ for all real x, which solves Cauchy's equation.

If f is not assumed continuous, by the axiom of choice there exists a basis $\{e_i\}_{i \in I}$ for the real numbers over the rationals. Thus any real number r has a unique representation as

$$r = q_1 e_{i_1} + \ldots + q_n e_{i_n}$$

where $q_1, \ldots, q_n \in \mathbb{Q}$. Any assignment of $f(e_i)$ on basis elements can be extended by rational-linearity to a function on $\mathbb{R}$ which satisfies Cauchy's equation.

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nonzero function, satisfying the equation $f(x+y) = f(x)f(y)$. Prove that there exists $c > 0$ such that $f(x) = c^x$ for all $x \in \mathbb{R}$.

Problem 2. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = f(x) + f(y) + f(x)f(y).$$

Problem 3. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = \frac{f(x) + f(y)}{1 + f(x)f(y)}.$$

Problem 4. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$f(xy) = xf(y) + yf(x).$$

Problem 5. Do there exist continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^3$ for all $x \in \mathbb{R}$.

Problem 6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 - y^2) = (x-y)(f(x) + f(y)).$$

Problem 7. Find all complex-valued functions of a complex variable satisfying

$$f(z) + zf(1-z) = 1 + z.$$

Problem 8. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x^2 - 2$ for all real numbers x?

Problem 9. Find all functions $f : (0, \infty) \to (0, \infty)$ subject to the conditions

i. $f(f(f(x))) + 2x = f(3x)$, for all $x > 0$.
ii. $\lim_{x \to \infty}(f(x) - x) = 0$.

Problem 10. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation

$$g(x - y) = g(x)g(y) + f(x)f(y)$$

for x and y in $\mathbb{R}$, and that $f(t) = 1$ and $g(t) = 0$ for some $t \neq 0$. Prove that f and g satisfy

$$g(x + y) = g(x)g(y) - f(x)f(y)$$

and

$$f(x \pm y) = f(x)g(y) \pm g(x)f(y)$$

for all real x and y.