COMBINATORICS

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The Polya-Redfield method

Based on the book *Combinatorics* by Daniel Marcus.

Given a string $x_1x_2x_3...x_n$, a rotation of the string has the form $x_{i+1}x_{i+2}x_{i+3}...x_nx_1x_2...x_i$. Given a string $x_1x_2...x_n$, the set of all strings obtained by rotating the string are called an *orbit* since rotating n times returns to the original string.

Problem 1. Consider strings of length 4 on letters A and B. Find all of the orbits.

Problem 2. Find the orbits of strings $AAABBB$, $AABAAB$ and $ABABAB$. For each string, how many rotations leave the string in its original position? This set of rotations is called the *stabilizer*.

Problem 3. Let R_i denote rotation by i places,

$$R_i(x_1x_2...x_n) = x_{i+1}x_{i+2}...x_nx_1x_2...x_i. \tag{1}$$

Prove that $R_iR_j = R_{i+j \bmod n}$.

Problem 4. Given a string of length n, prove that the size of its orbit times the size of its stabilizer is equal to n.

Problem 5. Consider all strings of length n on m letters. Prove that the sum of the size of the stabilizers is equal to n times the number of orbits under rotation.

Problem 6. The invariant number r_i of rotation R_i is the number of strings fixed by the rotation. Prove

$$\sum_{i=1}^{n} r_i = n\#\{\text{Orbits}\}.$$

Problem 7. Find the number of orbits on 12 letter strings that use the letters A, B.

Problem 8. Find all colorings of a 2×2 square with 3 colors if the square can be both rotated and reflected.

Problem 9. How many ways can five beads be arranged on a circular bracelet if each bead can be one of three colors. The bracelet can be rotated or flipped.

Problem 10. A 3×3 square is colored with the two colors A and B, and is allowed to be rotated but not flipped. Find the invariant numbers of the rotations by 0, 90, 180 and 270 degrees and calculate the number of colorings.

Explain why the coefficient on A^xB^{9-x} in the *pattern inventory*

$$\frac{(A+B)^9 + 2(A+B)(A^4+B^4)^2 + (A+B)(A^2+B^2)^4}{4} \tag{2}$$

gives the number of colorings with x A's and $9-x$ B's.

If the positions in the square are numbered

1	2	3
4	5	6
7	8	9

a 90 degree counterclockwise rotation maps 1 to 7, 7 to 9, 9 to 3 and 3 to 1, 2 to 4, 4 to 8, 8 to 6 and 6 to 2, and 5 to 5. This can be represented using a *cycle code* as $(1,7,9,3)(2,4,8,6)(5)$. The 180 degree rotation is represented by $(1,9)(7,3)(2,8)(4,6)(5)$. The first map is represented by $X_1X_4^2$ and the second by $X_1X_2^4$, since the first has one cycle of length 1 and 2 cycles of length 4, while the second has one cycle of length 1 and four cycles of length 2. The *cycle index polynomial* of the set of rotations is the average of the cycle codes,

$$\frac{X_1^4 + 2X_1X_4^2 + X_1X_2^4}{4}.$$

Problem 11. Prove that the *pattern inventory* on k colors is the result of substituting $X_i = A_1^i + ... + A_k^i$ into the cycle index polynomial.

Problem 12. There are 24 different orientations of a cube. Find the number of ways of coloring the cube using k colors.

Problem 13. The are 12 different orientations of a regular tetrahedron. Find the number of ways of coloring the faces using k colors.

Miscellaneous combinatorics problems

Problem 14. Prove that every graph has two vertices that are endpoints of the same number of edges.

Problem 15. Consider the sequence of functions and sets

$$\cdots \to A_n \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_{n-2}} A_{n-2} \xrightarrow{f_{n-3}} \cdots \xrightarrow{f_3} A_3 \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1.$$

Prove that if the sets A_n are non-empty and finite for all n, then there exists a sequence of elements $x_n \in A_n$, $n = 1, 2, 3, ...$, with the property that $f_n(x_{n+1}) = x_n$ for all $n \geqslant 1$.

Problem 16. For each permutation $a_1, a_2, ..., a_{10}$ of the integers $1, 2, 3, ..., 10$, form the sum

$$|a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8| + |a_9 - a_{10}|.$$

Find the average value of all such sums.

Problem 17. Let $f(n)$ be the number of permutations $a_1, a_2, ..., a_n$ of the integers $1, 2, ..., n$ such that (i) $a_1 = 1$ and (ii) $|a_i - a_{i+1}| \leqslant 2$, $i = 1, 2, ..., n-1$. Determine whether $f(1996)$ is divisible by 3.

Problem 18. Consider the sequences of real numbers $x_1 > x_2 > \cdots > x_n$ and $y_1 > y_2 > \cdots > y_n$, and let σ be a nontrivial permutation of the set $\{1, 2, ..., n\}$. Prove that

$$\sum_{i=1}^{n}(x_i - y_i)^2 < \sum_{i=1}^{n}(x_i - y_{\sigma(i)})^2.$$

Problem 19. Let $a_1, a_2, ..., a_n$ be a permutation of the numbers $1, 2, ..., n$. We call a_i a *large* integer if $a_i > a_j$ for all $i < j < n$. Find the average number of large integers over all permutations of the first n positive integers.

Problem 20. Let n be an odd integer greater than 1. Find the number of permutations σ of the set $\{1, 2, ..., n\}$ for which

$$|\sigma(1) - 1| + |\sigma(2) - 2| + \cdots + |\sigma(n) - n| = \frac{n^2 - 1}{2}.$$

Problem 21. A circle of radius 1 rolls without slipping on the outside of a circle of radius $\sqrt{2}$. The contact point of the circles in the initial position is colored. Any time a point of one circle touches a colored point of the other, it becomes itself colored. How many colored points will the moving circle have after 100 revolutions?

Problem 22. Prove that

$$\binom{2k}{k} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (2\sin\theta)^{2k} d\theta.$$

Problem 23. Let $(F_n)_n$ be the Fibonacci sequence, $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$. Prove that for any positive integer n,

$$F_1\binom{n}{1} + F_2\binom{n}{2} + \cdots + F_n\binom{n}{n} = F_{2n}.$$

Problem 24. Denote by $P(n)$ the number of partitions of the positive integer n, i.e., the number of ways of writing n as a sum of positive integers. Prove that the generating function of $P(n)$, $n \geqslant 1$, is given by

$$\sum_{n=0}^{\infty} P(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}$$

with the convention $P(0) = 1$.

Problem 25. Prove that the number of ways of writing n as a sum of distinct positive integers is equal to the number of ways of writing n as the sum of odd positive integers.

Problem 26. Prove that the number of non-negative integer solutions to the equation

$$x_1 + x_2 + \cdots + x_m = n$$

is equal to $\binom{m+n-1}{m-1}$.

Problem 27. Prove that the set of numbers $\{1, 2, ..., 2005\}$ can be colored with two colors such that any of its 18-term arithmetic sequences contains both colors.

Problem 28. A permutation σ of a set S is called a derangement if it does not have fixed points, i.e., if $\sigma(x) \neq x$ for all $x \in S$. Find the number of derangements of the set $\{1, 2, ..., n\}$.

Problem 29. An *exact covering system of congruences* is a collection of arithmetic progressions, $0 \leqslant a_i < m_i$,

$$a_i + m_i k, \qquad k \in \mathbb{Z}$$

that are disjoint, and whose union is the integers. For instance, every integer satisfies exactly one of the congruences 0 mod 2, 1 mod 4, 3 mod 12, 7 mod 12, 11 mod 24, 23 mod 24.

a. Prove the generating function identity

$$\frac{1}{1-x} = \sum_i \frac{x^{a_i}}{1 - x^{m_i}}.$$

b. Prove that every exact covering system of congruences with more than one congruence has a repeated step (e.g. 12 and 24 above).

Problem 30. Prove that $\sum_{m=0}^{\infty} \binom{2m}{m} \left(\frac{x}{2}\right)^{2m} = \frac{1}{\sqrt{1-x^2}}$.

Problem 31. A *permutation* σ of n letters is a one-to-one map of the numbers $\{1, 2, ..., n\}$. Permutations can be thought of as orderings of a deck of n cards. A *cycle* in a permutation is a sequence $a_1, a_2, ..., a_k$ such that $\sigma(a_1) = a_2, \sigma(a_2) = a_3, ..., \sigma(a_k) = a_1$. Every permutation can be split into disjoint cycles.

Given a set of numbers $b_1, b_2, b_3, ..., b_k$ with $b_1 + 2b_2 + 3b_3 + \cdots + kb_k = n$, let $c(b)$ denote the number of permutations of n numbers with b_1 cycles of length 1, b_2 cycles of length 2, ..., b_k cycles of length k. Let

$$C(x_1, x_2, ...; t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{b_1+2b_2+...+nb_n=n} c(b) x_1^{b_1} ... x_n^{b_n}.$$

This is the 'grand' cycle index generating function.

a. Prove that

$$c(b) = \frac{n!}{\prod_{j\geqslant 1} (b_j!) j^{b_j}}.$$

b. Using the above or otherwise, prove

$$C(x_1, x_2, ...; t) = \exp\left(\sum_{j\geqslant 1} \frac{x_j t^j}{j}\right).$$