ALGEBRA, LINEAR ALGEBRA

ROBERT HOUGH

One can check by induction that the Fibonacci sequence $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ can be obtained as the matrix power $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$. By diagonalizing the real symmetric matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ you may obtain a well-known formula for the nth Fibonacci number in terms of the Golden mean $\phi = \frac{1+\sqrt{5}}{2}$.

The van der Monde polynomial of n variables $x_1, ..., x_n$ is

$$V(x_1, ..., x_n) = \det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$

From the determinant formula

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{1,\sigma(1)} ... A_{n,\sigma(n)}$$

it follows that V is a polynomial of degree $1 + 2 + \cdots + n - 1 = \frac{(n-1)n}{2}$. By subtracting column i from column j, and noting that $x_j^k - x_i^k$ has a factor of $x_j - x_i$, conclude that $V(x_1, ..., x_n)$ is divisible by $\prod_{i<j}(x_j - x_i)$. Hence $V(x_1, ..., x_n) = c\prod_{i<j}(x_j - x_i)$. We have $c = 1$ by matching the coefficient on $x_2 x_3^2 x_4^3 \cdots x_n^{n-1}$.

Over $\mathbb{C}$, any matrix $n \times n$ matrix A is similar to an upper triangular matrix in *Jordan normal form*: for some invertible $n \times n$ matrix C,

$$C^{-1}AC = \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ 0 & 0 & J_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_k \end{pmatrix}.$$

Each J_i is called a *Jordan block* and has the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}.$$

Here λ_i is a generalized eigenvalue. The characteristic polynomial of A is $\det(\lambda I - A) = \prod_i (\lambda - \lambda_i)^{\dim J_i}$.

The trace of an $n \times n$ matrix A is $\operatorname{tr} A = \sum_i A_{ii}$. This satisfies $\operatorname{tr} AB = \operatorname{tr} BA$, and is the sum of the generalized eigenvalues counted with multiplicity.

Problem 1. Let M be an $n \times n$ complex matrix. Prove that there exist Hermitian matrices A and B such that $M = A + iB$. (A matrix X is called Hermitian if $\overline{X^t} = X$).

Problem 2. Do there exist $n \times n$ matrices A and B such $AB - BA = I_n$.

Problem 3. Compute the nth power of the $n \times n$ matrix $J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$.

Problem 4. Let $(F_n)_n$ be the Fibonacci sequence. Prove that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, for all $n \geqslant 1$.

Problem 5. Let $0 < p < m$. Prove $\det \begin{pmatrix} \binom{m}{0} & \binom{m}{1} & \cdots & \binom{m}{p} \\ \binom{m+1}{0} & \binom{m+1}{1} & \cdots & \binom{m+1}{p} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{m+p}{0} & \binom{m+p}{1} & \cdots & \binom{m+p}{p} \end{pmatrix} = 1.$

Problem 6. Compute the determinant of the $n \times n$ matrix $A = (a_{ij})_{ij}$, where

$$a_{ij} = \begin{cases} (-1)^{|i-j|} & i \neq j \\ 2 & i = j \end{cases}.$$

Problem 7. Prove the formula for the determinant of a circulant matrix:

$$\det \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_3 & x_4 & x_5 & \cdots & x_2 \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{pmatrix} = \prod_{j=0}^{n-1} \left(\sum_{k=1}^{n} \zeta^{jk} x_k \right),$$

where $\zeta = e^{2\pi i/n}$.

Problem 8. Let $\alpha = \frac{\pi}{n+1}$, $n > 2$. Prove that the $n \times n$ matrix

$$\begin{pmatrix} \sin\alpha & \sin 2\alpha & \cdots & \sin n\alpha \\ \sin 2\alpha & \sin 4\alpha & \cdots & \sin 2n\alpha \\ \vdots & \vdots & \ddots & \vdots \\ \sin n\alpha & \sin 2n\alpha & \cdots & \sin n^2\alpha \end{pmatrix}$$

is invertible.

Problem 9. Denote by $M_n(\mathbb{R})$ the set of $n \times n$ matrices with real entries and let $f : M_n(\mathbb{R}) \to \mathbb{R}$ be a linear function. Prove that there exists a unique matrix $C \in M_n(\mathbb{R})$ such that $f(A) = \operatorname{tr}(AC)$ for all $A \in M_n(\mathbb{R})$. In addition, if $f(AB) = f(BA)$ for all matrices A and B, prove that there exists $\lambda \in \mathbb{R}$ such that $f(A) = \lambda \operatorname{tr} A$ for any A.

Problem 10. For an $n \times n$ matrix A denote by $\phi_k(A)$ the symmetric polynomial in the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of A, $\phi_k(A) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_k}$, $k = 1, 2, ..., n$. Thus $\phi_1(A)$ is the trace and $\phi_n(A)$ is the determinant. Prove that for two $n \times n$ matrices A and B, $\phi_k(AB) = \phi_k(BA)$.

Problem 11. Assume that a and b are elements of a group with identity element e satisfying $(aba^{-1})^n = e$ for some positive integer n. Prove that $b^n = e$.

Problem 12. Given Γ a finite multiplicative group of matrices with complex entries, denote by M the sum of the matrices in Γ. Prove that the determinant of M is an integer.

Problem 13. Let $\{k_1, k_2, \ldots, k_m\}$ be a set of integers. Show that there is an $m \times m$ matrix A with integers coefficients such that the matrices $A + k_j I$, for $1 \leqslant j \leqslant m$, are invertible and their inverses also have integer coefficients. (I is the $m \times m$ identity matrix).