Math 639: Lecture 9

Recurrence, Renewal theory

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Recurrent and possible values

Definition

Let $X_1, X_2, \ldots$ be i.i.d. in $\mathbb{R}^d$ and let $S_n = X_1 + \cdots + X_n$. The number $x \in \mathbb{R}^d$ is said to be a *recurrent value* for the random walk S_n if for every $\epsilon > 0$,

$$\mathrm{Prob}(\|S_n - x\|_\infty < \epsilon \text{ i.o.}) = 1.$$

A number x is called a *possible value* of the random walk if for any $\epsilon > 0$, there is an n such that

$$\mathrm{Prob}(\|S_n - x\|_\infty < \epsilon) > 0.$$

Recurrent and possible values

Theorem

The set V of recurrent values is either $\varnothing$ or a closed subgroup of $\mathbb{R}^d$. In the second case $V = U$, the set of possible values.

Recurrent and possible values

Proof.

- Suppose $V \neq \emptyset$. Since V^c is open, V is closed.
- We prove: if $x \in U$ and $y \in V$ then $y - x \in V$.
- Let $p_{\delta,m}(z) = \mathrm{Prob}(\|S_n - z\|_\infty \geqslant \delta$ for all $n \geqslant m)$. If $y - x \notin V$, there is an $\epsilon > 0$ and $m \geqslant 1$ so that $p_{2\epsilon,m}(y-x) > 0$.
- Choose k so that $\mathrm{Prob}(\|S_k - x\|_\infty < \epsilon) > 0$.
- Note that

$$\mathrm{Prob}(\|S_n - S_k - (y-x)\|_\infty \geqslant 2\epsilon \text{ for all } n \geqslant k+m) = p_{2\epsilon,m}(y-x)$$

and is independent of $\{\|S_k - x\|_\infty < \epsilon\}$. Thus

$$p_{\epsilon,m+k}(y) \geqslant \mathrm{Prob}(\|S_k - x\|_\infty < \epsilon) p_{2\epsilon,m}(y-x) > 0,$$

which contradicts $y \in V$. Hence $y - x \in V$.

□

Recurrent and possible values

Proof.

The above demonstrates that V is a closed subgroup, hence contains 0, and thus is equal to U. $\square$

Transience and recurrence

Definition

If $V \neq 0$ the random walk is *transient*, otherwise *recurrent*. The *return times to 0* are defined by

$$\tau_0 = 0, \qquad \tau_n = \inf\{m > \tau_{n-1} : S_m = 0\}, \; n \geqslant 1.$$

As mentioned last lecture, $\text{Prob}(\tau_n < \infty) = \text{Prob}(\tau_1 < \infty)^n$.

Transience and recurrence

Theorem

For any random walk, the following are equivalent.

1. $\mathrm{Prob}(\tau_1 < \infty) = 1$
2. $\mathrm{Prob}(S_m = 0 \text{ } i.o.) = 1$
3. $\sum_{m=0}^{\infty} \mathrm{Prob}(S_m = 0) = \infty.$

Transience and recurrence

Proof.

- If $\mathrm{Prob}(\tau_1 < \infty) = 1$, then $\mathrm{Prob}(\tau_n < \infty) = 1$ for all n and $\mathrm{Prob}(S_m = 0 \text{ i.o.}) = 1$, so 1 implies 2.
- 2 implies 3 follows from Borel-Cantelli.
- Let
$$V = \sum_{m=0}^{\infty} \mathbf{1}_{(S_m=0)} = \sum_{n=0}^{\infty} \mathbf{1}_{(\tau_n<\infty)}$$
and calculate to give 3 implies 1,
$$\begin{aligned} \mathrm{E}[V] &= \sum_{m=0}^{\infty} \mathrm{Prob}(S_m = 0) = \sum_{n=0}^{\infty} \mathrm{Prob}(\tau_n < \infty) \\ &= \sum_{n=0}^{\infty} \mathrm{Prob}(\tau_1 < \infty)^n = \frac{1}{1 - \mathrm{Prob}(\tau_1 < \infty)}. \end{aligned}$$

$\square$

Transience and recurrence

Definition

Simple random walk in $\mathbb{R}^d$ is defined by letting steps satisfy

$$\mathrm{Prob}(X_i = e_j) = \mathrm{Prob}(X_i = -e_j) = \frac{1}{2d}.$$

Theorem

Simple random walk is recurrent in $d \leqslant 2$ and transcient in $d \geqslant 3$.

Transience and recurrence

Proof.

- Let $\rho_d(m) = \mathrm{Prob}(S_m = 0)$ in dimension d. This is 0 by parity considerations if m is odd.
- We have $\rho_1(2n) \sim (\pi n)^{-\frac{1}{2}}$ as $n \to \infty$, which proves the recurrence in dimension 1.
- In dimension 2, let T_n^1 and T_n^2 be independent one dimensional simple random walks. The walk (T_n^1, T_n^2) takes steps, with equal probability $(1,1), (1,-1), (-1,1), (-1,-1)$. Rotating by 45 degrees and dividing by $\sqrt{2}$ gives S_n. Hence $\rho_2(2n) = \rho_1(2n)^2 \sim \frac{1}{\pi n}$. Since $\sum_n \frac{1}{n}$ diverges, the walk is recurrent.

$\square$

Transience and recurrence

Proof.

- Estimate

$$\begin{aligned}
\rho_3(2n) &= 6^{-2n} \sum_{j,k} \frac{(2n)!}{(j!k!(n-j-k)!)^2} \\
&= 2^{-2n} \binom{2n}{n} \sum_{j,k} \left(3^{-n} \frac{n!}{j!k!(n-j-k)!}\right)^2 \\
&\leqslant 2^{-2n} \binom{2n}{n} \max_{j,k} 3^{-n} \frac{n!}{j!k!(n-j-k)!}.
\end{aligned}$$

- The maximum occurs for $j, k, n-j-k$ all at least $\left\lfloor \frac{n}{3} \right\rfloor$ and the estimate $\rho_3(2n) \ll \frac{1}{n^{\frac{3}{2}}}$ follows from Stirling's formula. Since this is summable, the transience follows.
- Transience for $d > 3$ follows by projecting on the first 3 coordinates.

□

We now consider more general random walks.

Lemma

If $\sum_{n=1}^{\infty} \mathrm{Prob}(\|S_n\|_\infty < \epsilon) < \infty$, *then* $\mathrm{Prob}(\|S_n\|_\infty < \epsilon \text{ i.o.}) = 0$. *If* $\sum_{n=1}^{\infty} \mathrm{Prob}(\|S_n\|_\infty < \epsilon) = \infty$ *then* $\mathrm{Prob}(\|S_n\|_\infty < 2\epsilon \text{ i.o.}) = 1$.

Transience and recurrence

Proof.

- The first conclusion follows from Borel-Cantelli.
- Let $F = \{\|S_n\| < \epsilon \text{ i.o.}\}^c$. Calculate

$$\begin{aligned}\mathrm{Prob}(F) &= \sum_{m=0}^{\infty} \mathrm{Prob}(\|S_m\|_\infty < \epsilon, \|S_n\|_\infty \geqslant \epsilon \text{ for all } n \geqslant m+1)\\ &\geqslant \sum_{m=0}^{\infty} \mathrm{Prob}(\|S_m\|_\infty < \epsilon, \|S_n - S_m\|_\infty \geqslant 2\epsilon \text{ for all } n \geqslant m+1)\\ &= \sum_{m=0}^{\infty} \mathrm{Prob}(\|S_m\|_\infty < \epsilon)\rho_{2\epsilon,1}\end{aligned}$$

where $\rho_{\delta,k} = \mathrm{Prob}(\|S_n\|_\infty \geqslant \delta \text{ for all } n \geqslant k)$. Since $\sum_{m=0}^{\infty} \mathrm{Prob}(\|S_m\|_\infty < \epsilon) = \infty$, $\rho_{2\epsilon,1} = 0$.

□

Transience and recurrence

Proof.

- Let
$$A_m = \{\|S_m\|_\infty < \epsilon, \|S_n\|_\infty \geqslant \epsilon \text{ for all } n \geqslant m + k\}.$$
Since any ω belongs to at most k A_m,
$$k \geqslant \sum_{m=0}^{\infty} \mathrm{Prob}(A_m) \geqslant \sum_{m=0}^{\infty} \mathrm{Prob}\left(\|S_m\|_\infty < \epsilon\right) \rho_{2\epsilon,k}.$$
- Thus $\rho_{2\epsilon,k} = \mathrm{Prob}(\|S_j\|_\infty \geqslant 2\epsilon \text{ for all } j \geqslant k) = 0$ for each k.

□

Transience and recurrence

Lemma

Let m be an integer $\geqslant 2$.

$$\sum_{n=0}^{\infty} \mathrm{Prob}(\|S_n\|_\infty < m\epsilon) \leqslant (2m)^d \sum_{n=0}^{\infty} \mathrm{Prob}(\|S_n\|_\infty < \epsilon).$$

Transience and recurrence

Proof.

- Write

$$\sum_{n=0}^{\infty} \mathrm{Prob}(\|S_n\|_\infty < m\epsilon) \leqslant \sum_{n=0}^{\infty} \sum_{k} \mathrm{Prob}(S_n \in k\epsilon + [0, \epsilon)^d).$$

The inner sum is over $k \in \{-m, \ldots, m-1\}^d$.

$\square$

Proof.

- Let $T_k = \inf\{\ell \geqslant 0 : S_\ell \in k\epsilon + [0,\epsilon)^d\}$. Thus

$$\sum_{n=0}^{\infty} \mathrm{Prob}(S_n \in k\epsilon + [0,\epsilon)^d) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{n} \mathrm{Prob}(S_n \in k\epsilon + [0,\epsilon)^d, T_k = \ell)$$

$$\leqslant \sum_{\ell=0}^{\infty}\sum_{n=\ell}^{\infty} \mathrm{Prob}(\|S_n - S_\ell\|_\infty < \epsilon, T_k = \ell)$$

$$= \sum_{m=0}^{\infty} \mathrm{Prob}(T_k = m) \sum_{j=0}^{\infty} \mathrm{Prob}(\|S_j\| < \epsilon) \leqslant \sum_{j=0}^{\infty} \mathrm{Prob}(\|S_j\|_\infty < \epsilon).$$

The proof is complete since there are $(2m)^d$ values of k.

$\square$

Transience and recurrence

Theorem

The convergence (resp. divergence) of $\sum_n \mathrm{Prob}(\|S_n\|_\infty < \epsilon)$ *for a single value of* $\epsilon > 0$ *is sufficient for transience (resp. recurrence).*

If $d = 1$, if $\mathrm{E}[X_i] = \mu \neq 0$, then the strong law of large numbers implies $S_n/n \to \mu$, so $|S_n| \to \infty$ and S_n is transient.

The Chung-Fuchs Theorem

Theorem (Chung-Fuchs theorem)

Suppose $d = 1$. If the weak law of large numbers holds in the form $S_n/n \to 0$ in probability, then S_n is recurrent.

The Chung-Fuchs Theorem

Proof.

- Let $u_n(x) = \text{Prob}(|S_n| < x)$ for $x > 0$.
- Applying the previous lemma,

$$\sum_{n=0}^{\infty} u_n(1) \geqslant \frac{1}{2m} \sum_{n=0}^{\infty} u_n(m) \geqslant \frac{1}{2m} \sum_{n=0}^{Am} u_n(n/A)$$

  for any $A < \infty$ since $u_n(x) \geqslant 0$ and is increasing in x.
- Since $u_n(n/A) \to 1$, letting $m \to \infty$ gives

$$\sum_{n=0}^{\infty} u_n(1) \geqslant A/2.$$

  for all A. The conclusion now follows from the previous theorem.

$\square$

Transience and recurrence

Theorem

If S_n is a random walk in $\mathbb{R}^2$ and $\frac{S_n}{n^{\frac{1}{2}}}$ converges to a non-degenerate normal distribution, then S_n is recurrent.

Transience and recurrence

Proof.

- Let $u(n, m) = \text{Prob}(\|S_n\|_\infty < m)$.
- We have
$$\sum_{n=0}^{\infty} u(n, 1) \geqslant (4m^2)^{-1} \sum_{n=0}^{\infty} u(n, m).$$
- If $m/\sqrt{n} \to c$, then
$$u(n, m) \to \int_{[-c,c]^2} n(x) dx$$
where $n(x)$ is the limiting normal distribution.
- Let $u([\theta m^2], m) \to \rho(\theta^{-\frac{1}{2}})$.

□

Transience and recurrence

Proof.

- Write

$$\frac{1}{m^2}\sum_{n=0}^{\infty} u(n,m) = \int_0^{\infty} u(\lfloor \theta m^2 \rfloor, m)\, d\theta$$

and let $m \to \infty$ to obtain

$$\liminf_{m\to\infty} \frac{1}{4m^2}\sum_{n=0}^{\infty} u(n,m) \geqslant \frac{1}{4}\int_0^{\infty} \rho(\theta^{-\frac{1}{2}})\, d\theta.$$

The integral diverges since $\rho(c) = \int_{[-c,c]^2} n(x)dx \sim n(0)(2c)^2$ as $c \downarrow 0$.

$\square$

Transience and recurrence

Theorem

Let $\phi(t)$ be the characteristic function of X_i. Let $\delta > 0$. S_n is recurrent if and only if

$$\sup_{r<1} \int_{(-\delta,\delta)^d} \Re \frac{1}{1 - r\phi(y)} dy = \infty.$$

Parseval

Theorem (Parseval relation)

Let μ and ν be probability measures on $\mathbb{R}^d$ with characteristic functions ϕ and ψ. Then

$$\int \psi(t)\mu(dt) = \int \phi(x)\nu(dx).$$

Proof.

By Fubini,

$$\begin{aligned}\int \psi(t)\mu(dt) &= \int\int e^{itx}\nu(dx)\mu(dt)\\ &= \int\int e^{itx}\mu(dt)\nu(dx) = \int \phi(x)\nu(dx).\end{aligned}$$

□

Transience and recurrence

Lemma

If $|x| \leqslant \frac{\pi}{3}$ *then* $1 - \cos x \geqslant \frac{x^2}{4}$.

Proof.

If $|z| \leqslant \frac{\pi}{3}$ then $\cos z \geqslant \frac{1}{2}$. Hence

$$\sin y = \int_0^y \cos z dz \geqslant \frac{y}{2}$$

$$1 - \cos x = \int_0^x \sin y dy \geqslant \int_0^x \frac{y}{2} dy = \frac{x^2}{4}.$$

□

Proof of Recurrence Theorem.

- The density
$$F_\delta(x) = \frac{\delta - |x|}{\delta^2}\mathbf{1}(|x| \leqslant \delta)$$
has characteristic function $\hat{F}_\delta(t) = 2\frac{1-\cos\delta t}{(\delta t)^2}$.
- Let S_n have density μ_n. One has
$$\begin{aligned}\mathrm{Prob}\left(\|S_n\|_\infty \leqslant \frac{1}{\delta}\right) &\leqslant 4^d \int \prod_{i=1}^d \frac{1-\cos(\delta t_i)}{(\delta t_i)^2}\mu_n(t) \\ &= 2^d \int_{(-\delta,\delta)^d} \prod_{i=1}^d \frac{\delta - |x_i|}{\delta^2}\phi^n(x)dx.\end{aligned}$$

$\square$

Transience and recurrence

Proof of Recurrence Theorem.

- Hence

$$\sum_{n=0}^{\infty} r^n \operatorname{Prob}\left(\|S_n\|_\infty < \frac{1}{\delta}\right) \leqslant 2^d \int_{(-\delta,\delta)^d} \prod_{i=1}^{d} \frac{\delta - |x_i|}{\delta^2} \frac{1}{1 - r\phi(x)} dx$$

and

$$\sum_{n=0}^{\infty} \operatorname{Prob}\left(\|S_n\|_\infty < \frac{1}{\delta}\right) \leqslant \left(\frac{2}{\delta}\right)^d \sup_{r<1} \int_{(-\delta,\delta)^d} \Re \frac{1}{1 - r\phi(x)} dx.$$

Thus finiteness of the right hand side gives transience of the walk.

□

Transience and recurrence

Proof of Recurrence Theorem.

- For the reverse direction, use density $G_\delta(x) = \frac{\delta\left(1-\cos\left(\frac{x}{\delta}\right)\right)}{\pi x^2}$, with characteristic function $\hat{G}_\delta(t) = (1 - |\delta t|)\mathbf{1}(|t| \leqslant \frac{1}{\delta})$.
- Hence

$$\text{Prob}\left(\|S_n\|_\infty < \frac{1}{\delta}\right) \geqslant \int_{(-1/\delta,1/\delta)^d} \prod_{i=1}^d (1 - |\delta x_i|)\mu_n(dx)$$
$$= \int \prod_{i=1}^d \frac{\delta(1 - \cos(t_i/\delta))}{\pi t_i^2} \phi^n(t) dt.$$

□

Proof of Recurrence Theorem.

- Hence

$$\sum_{n=0}^{\infty} r^n \operatorname{Prob}(\|S_n\|_\infty < 1/\delta) \geqslant \int \prod_{i=1}^{d} \frac{\delta(1-\cos(t_i/\delta))}{\pi t_i^2} \frac{1}{1-r\phi(t)} dt$$
$$\geqslant (4\pi\delta)^{-d} \int_{(-\delta,\delta)^d} \Re \frac{1}{1-r\phi(t)} dt.$$

- Letting $r \uparrow 1$ proves the theorem.

□

Transience and recurrence

Definition

A random walk in $\mathbb{R}^3$ is *truly three-dimensional* if the distribution of X_1 has $\mathrm{Prob}(X_1 \cdot \theta \neq 0) > 0$ for all $\theta \neq 0$.

Theorem

No truly three-dimensional random walk is recurrent.

Transience and recurrence

Proof.

- If $z = a + bi$ with $a \leqslant 1$,

$$\Re \frac{1}{1-z} = \frac{1-a}{(1-a)^2 + b^2} \leqslant \frac{1}{1-a}.$$

- Hence

$$\Re \frac{1}{1 - r\phi(t)} \leqslant \frac{1}{\Re(1 - r\phi(t))} \leqslant \frac{1}{\Re(1 - \phi(t))}.$$

- Estimate

$$\Re(1 - \phi(t)) = \int (1 - \cos(xt))\mu(dx) \geqslant \int_{|x \cdot t| < \frac{\pi}{3}} \frac{|x \cdot t|^2}{4} \mu(dx).$$

$\square$

Transience and recurrence

Proof.

- Let $t = \rho\theta$ where $\theta \in S = \{x : |x| = 1\}$. This gives

$$\Re(1 - \phi(\rho\theta)) \geqslant \frac{\rho^2}{4} \int_{|x\cdot\theta| < \frac{\pi}{3\rho}} |x \cdot \theta|^2 \mu(dx).$$

- Letting $\rho \to 0$ and $\theta(\rho) \to \theta$,

$$\liminf_{\rho \to 0} \int_{|x\cdot\theta(\rho)| < \frac{\pi}{3\rho}} |x \cdot \theta(\rho)|^2 \mu(dx) \geqslant \int |x \cdot \theta|^2 \mu(dx) > 0.$$

- This implies that for $\rho < \rho_0$

$$\inf_{\theta \in S} \int_{|x\cdot\theta| < \frac{\pi}{3\rho}} |x \cdot \theta|^2 \mu(dx) = C > 0.$$

$\square$

Transience and recurrence

Proof.

- It follows that for $0 < \rho < \rho_0$, $\Re(1 - \phi(\rho\theta)) \geqslant \frac{C\rho^2}{4}$.
- Thus

$$\int_{(-\delta,\delta)^d} \Re \frac{1}{1 - r\phi(y)} dy \leqslant \int_0^{\delta\sqrt{d}} \rho^{d-1} d\rho \int \frac{1}{\Re(1 - \phi(\rho\theta))} d\theta$$
$$\leqslant C' \int_0^1 \rho^{d-3} d\rho < \infty.$$

$\square$

Paths

Definition

Consider simple random walk on $\mathbb{Z}$. A polygonal line has segments $(k-1, S_{k-1}) \to (k, S_k)$. A *path* is a polygonal line that is a possible outcome of simple random walk.

To count the number of paths from $(0,0)$ to (n,x), introduce $a = \frac{n+x}{2}$ and $b = \frac{n-x}{2}$. The number $N_{n,x}$ of paths is $\binom{n}{a}$.

The reflection principle

Theorem (Reflection principle)

If $x, y > 0$, then the number of paths from $(0, x)$ to (n, y) that are 0 at some time is equal to the number of paths from $(0, -x)$ to (n, y).

The reflection principle

Proof.

- Suppose $(0, s_0), (1, s_1), \ldots, (n, s_n)$ is a path from $(0, x)$ to (n, y).
- Let $K = \inf\{k : s_k = 0\}$. Let $s'_k = -s_k$ for $k \leqslant K$ and $s'_k = s_k$ for $k > K$. Thus (k, s'_k) is a path from $(0, -x)$ to (n, y).
- Conversely, if $(0, t_0), (1, t_1), \ldots, (n, t_n)$ is a path from $(0, -x)$ to (n, y), then it must cross 0. Set $K = \inf\{k : t_k = 0\}$ and let $t'_k = -t_k$ for $k \leqslant K$ and $t'_k = t_k$ for $k > K$.
- Thus (k, t'_k), $0 \leqslant k \leqslant n$ is a path from $(0, x) \to (n, y)$ that is 0 at time K. This completes the bijection.

$\square$

Ballot theorem

Theorem (Ballot theorem)

Suppose that in an election candidate A gets α votes and candidate B gets β votes, where $\beta < \alpha$. Given uniform ordering of the votes, the probability that throughout the counting A always leads B is $\frac{\alpha-\beta}{\alpha+\beta}$.

Ballot theorem

Proof.

- The number of admissible arrangements of the votes is the number of paths from $(1,1)$ to (n,x) that don't cross 0.
- By the reflection principle, the number of paths from $(1,1)$ to (n,x) which do cross 0 is equal to the number of paths from $(1,1)$ to $(n,-x)$.
- Hence, the number of admissible paths is

$$\begin{aligned} N_{n-1,x-1} - N_{n-1,x+1} &= \binom{n-1}{\alpha-1} - \binom{n-1}{\alpha} \\ &= \frac{(n-1)!}{(\alpha-1)!(n-\alpha)!} - \frac{(n-1)!}{\alpha!(n-\alpha-1)!} \\ &= \frac{\alpha-(n-\alpha)}{n}\frac{n!}{\alpha!(n-\alpha)!} = \frac{\alpha-\beta}{\alpha+\beta}N_{n,x}. \end{aligned}$$

□

Visits to 0

Lemma

$\mathrm{Prob}(S_1 \neq 0, S_2 \neq 0, ..., S_{2n} \neq 0) = \mathrm{Prob}(S_{2n} = 0).$

Visits to 0

Proof.

By the Ballot theorem

$$\begin{aligned}\mathrm{Prob}(S_1 > 0, \ldots, S_{2n} > 0) &= \sum_{r=1}^{\infty} \mathrm{Prob}(S_1 > 0, \ldots, S_{2n-1} > 0, S_{2n} = 2r) \\ &= \frac{1}{2^{2n}} \sum_{r=1}^{\infty} (N_{2n-1,2r-1} - N_{2n-1,2r+1}) = \frac{N_{2n-1,1}}{2^{2n}}.\end{aligned}$$

Since $\mathrm{Prob}(S_{2n-1} = 1) = \mathrm{Prob}(S_{2n} = 0)$ we obtain

$$\mathrm{Prob}(S_1 > 0, \ldots, S_{2n} > 0) = \frac{1}{2}\mathrm{Prob}(S_{2n} = 0)$$

the claim follows by symmetry. □

Visits to 0

Set $L_{2n} = \sup\{m \leqslant 2n : S_m = 0\}$.

Lemma

Let $u_{2m} = \mathrm{Prob}(S_{2m} = 0)$. *Then* $\mathrm{Prob}(L_{2n} = 2k) = u_{2k}u_{2n-2k}$.

Proof.

$\mathrm{Prob}(L_{2n} = 2k) = \mathrm{Prob}(S_{2k} = 0, S_{2k+1} \neq 0, ..., S_{2n} \neq 0) = u_{2k}u_{2n-2k}$. $\square$

The arcine law

Theorem

For $0 < a < b < 1$,

$$\mathrm{Prob}\left(a \leqslant \frac{L_{2n}}{2n} \leqslant b\right) \to \frac{1}{\pi}\int_a^b (x(1-x))^{-\frac{1}{2}}\,dx.$$

The arcsine law

Proof.

- Since $u_{2n} = \frac{\binom{2n}{n}}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}$ one obtains that if $\frac{k}{n} \to x$ as $n \to \infty$, then

$$n \operatorname{Prob}(L_{2n} = 2k) \to \frac{1}{\pi\sqrt{x(1-x)}}.$$

- The convergence is uniform on compact sets. Thus

$$\begin{aligned}\operatorname{Prob}\left(a \leqslant \frac{L_{2n}}{2n} \leqslant b\right) &= \sum_{2an \leqslant 2k \leqslant 2bn} \operatorname{Prob}(L_{2n} = 2k) \\ &\to \frac{1}{\pi}\int_a^b \frac{dx}{\sqrt{x(1-x)}}.\end{aligned}$$

□

The arcine law

Theorem

Let π_{2n} be the number of segments $(k-1, S_{k-1}) \to (k, S_k)$ that lie above the axis, i.e. in $\{(x,y) : y \geqslant 0\}$, and let $u_m = \mathrm{Prob}(S_m = 0)$.

$$\mathrm{Prob}(\pi_{2n} = 2k) = u_{2k} u_{2n-2k}$$

and consequently, if $0 < a < b < 1$,

$$\mathrm{Prob}\left(a \leqslant \frac{\pi_{2n}}{2n} \leqslant b\right) \to \frac{1}{\pi} \int_a^b \frac{dx}{\sqrt{x(1-x)}}.$$

The arcsine law

Proof.

- Let $\beta_{2k,2n} = \text{Prob}(\pi_{2n} = 2k)$. We prove $\beta_{2k,2n} = u_{2k}u_{2n-2k}$ by induction.
- When $n = 1$,
$$\beta_{0,2} = \beta_{2,2} = \frac{1}{2} = u_0 u_2.$$
- Calculate
$$\begin{aligned}
\frac{1}{2}u_{2n} &= \text{Prob}(S_1 > 0, S_2 > 0, ..., S_{2n} > 0)\\
&= \text{Prob}(S_1 = 1, S_2 - S_1 \geqslant 0, ..., S_{2n} - S_1 \geqslant 0)\\
&= \frac{1}{2}\text{Prob}(S_1 \geqslant 0, ..., S_{2n-1} \geqslant 0)\\
&= \frac{1}{2}\text{Prob}(S_1 \geqslant 0, ..., S_{2n} \geqslant 0) = \frac{1}{2}\beta_{2n,2n} = \frac{1}{2}\beta_{0,2n}.
\end{aligned}$$

□

The arcsine law

Proof.

- Let R be the time of the first return to 0, and set $f_{2m} = \mathrm{Prob}(R = 2m)$. We have

$$\beta_{2k,2n} = \frac{1}{2}\sum_{m=1}^{k} f_{2m}\beta_{2k-2m,2n-2m} + \frac{1}{2}\sum_{m=1}^{n-k} f_{2m}\beta_{2k,2n-2m}.$$

- By induction,

$$\beta_{2k,2n} = \frac{1}{2}u_{2n-2k}\sum_{m=1}^{k} f_{2m}u_{2k-2m} + \frac{1}{2}u_{2k}\sum_{m=1}^{n-k} f_{2m}u_{2n-2k-2m}.$$

The conclusion holds, since $u_{2k} = \sum_{m=1}^{k} f_{2m}u_{2k-2m}$.

□

Renewals

Let $\xi_1, \xi_2, \ldots$ be i.i.d. positive random variables with distribution F and define a sequence of times by

$$T_0 = 0, \qquad T_k = T_{k-1} + \xi_k, \; k \geqslant 1.$$

The T_k are referred to as *renewals*. Let $N_t = \inf\{k : T_k > t\}$. Define $U(t) = \mathrm{E}[N_t]$.

Theorem

As $t \to \infty$, $\frac{U(t)}{t} \to \frac{1}{\mu}$.

Renewals

Proof.

- Pick $\delta > 0$ so that $\mathrm{Prob}(\xi_i > \delta) = \epsilon > 0$. Pick K so that $K\delta \geqslant t$. Since K consecutive ξ_i's greater than δ make $T_n > t$,

$$\mathrm{Prob}(N_t > mK) \leqslant (1 - \epsilon^K)^m.$$

  Thus $\mathrm{E}[N_t] < \infty$.

- By Wald's equation,

$$\mu\, \mathrm{E}[N_t] = \mathrm{E}[T_{N_t}] \geqslant t,$$

  so $U(t) \geqslant \frac{t}{\mu}$.

□

Proof.

- If $\mathrm{Prob}(\xi_i \leqslant c) = 1$ then $\mu\,\mathrm{E}[N_t] = \mathrm{E}[T_{N_t}] \leqslant t + c$, so the result holds for bounded distributions. If we replace $\overline{\xi}_i = \min(\xi_i, c)$ and define $\overline{T}_n$ and $\overline{N}_t$ then

$$\mathrm{E}[N_t] \leqslant \mathrm{E}[\overline{N}_t] \leqslant \frac{t + c}{\mathrm{E}[\overline{\xi}_i]}.$$

Let $t \to \infty$, then $c \to \infty$ to obtain $\limsup_{t\to\infty} \frac{\mathrm{E}[N_t]}{t} \leqslant \frac{1}{\mu}$.

□

Renewal measure

Definition

The *renewal measure* of a process T_k is the measure

$$U(A) = \sum_{n=0}^{\infty} \text{Prob}(T_n \in A).$$

Theorem (Blackwell's renewal theorem)

If F is nonarithmetic with mean $\mu < \infty$, then $U([t, t+h]) \to \frac{h}{\mu}$ as $t \to \infty$.

See Durrett p.211 for the case $\mu = \infty$.

Delayed renewal process

Definition

If $T_0 \geqslant 0$ is independent of $\xi_1, \xi_2, \ldots$ and has distribution G, then $T_k = T_{k-1} + \xi_k$, $k \geqslant 1$ defines a *delayed renewal process*, and G is the *delay distribution*.

If we let $N_t = \inf\{k : T_k > t\}$ and set $V(t) = \mathrm{E}[N_t]$, then

$$V(t) = \int_0^t U(t-s)dG(s).$$

Similarly,

$$U(t) = 1 + \int_0^t U(t-s)dF(s).$$

or $U = \mathbf{1}_{[0,\infty)}(t) + U * F$, and $V = G * U = G + V * F$.

Stationary renewal process

Definition

When $G(t) = \frac{1}{\mu}\int_0^t 1 - F(y)dy$ and $V(t) = G(t) + \int_0^t \frac{t-y}{\mu} dF(y) = \frac{t}{\mu}$, the process $T_0, T_1, T_2, \ldots$ is called the *stationary renewal process* associated to ξ_i.

Blackwell's renewal theorem

Proof of Blackwell's theorem in case $\mu < \infty$.

- Let $T_0, T_1, T_2, \ldots$ be a renewal process, and let $T_0', T_1', T_2', \ldots$ be an independent stationary renewal process.
- Given $\epsilon > 0$, we find J and K such that $|T_J - T_K'| < \epsilon$.
- Let $\eta_1, \eta_2, \ldots$ and $\eta_1', \eta_2', \ldots$ be i.i.d. independent of T_n and T_n', taking values 0 and 1 with probability $\frac{1}{2}$.
- Let $\nu_n = \eta_1 + \cdots + \eta_n$ and $\nu_n' = 1 + \eta_1' + \cdots + \eta_n'$, $S_n = T_{\nu_n}$, and $S_n' = T'_{\nu_n'}$.

$\square$

Blackwell's renewal theorem

Proof of Blackwell's theorem in case $\mu < \infty$.

- The increments of $S_n - S_n'$ are 0 with probability $\frac{1}{4}$ and are symmetric about 0. Since ξ_k is nonarithmetic, $S_n - S_n'$ is irreducible. Since the increments have mean 0,

$$N = \inf\{n : |S_n - S_n'| < \epsilon\}$$

  has $\mathrm{Prob}(N < \infty) = 1$. Set $J = \nu_N$ and $K = \nu_N'$.
- Define *coupling*

$$T_n'' = \begin{cases} T_n & n \leqslant J \\ T_J + T_{K+(n-J)} - T_K' & n > J \end{cases}.$$

  Thus $T_{j+i}'' - T_J'' = T_{K+i}' - T_K'$ for $i \geqslant 1$.
- By construction, T_n and T_n'' have the same distribution.

□

Proof of Blackwell's theorem in case $\mu < \infty$.

- Let

$$N'(s,t) = |\{n : T_n' \in [s,t]\}|, \qquad N''(s,t) = |\{n : T_n'' \in [s,t]\}|.$$

We have

$$N''(t, t+h) = N'(t + T_K' - T_J, t + h + T_K' - T_J).$$

This is sandwiched between $N'(t+\epsilon, t+h-\epsilon)$ and $N'(t-\epsilon, t+h+\epsilon)$. Hence

$$\frac{h-2\epsilon}{\mu} - \mathrm{Prob}(T_J > t)U(h) \leqslant U([t, t+h]) \leqslant \frac{h+2\epsilon}{\mu} + \mathrm{Prob}(T_J > t)U(h).$$

□

Renewal equation

Definition

A *renewal equation* is an equation $H = h + H * F$.

Examples include $h \equiv 1$ and $U(t) = 1 + \int_0^t U(t-s)dF(s)$ and $h(t) = G(t)$, $V(t) = G(t) + \int_0^t V(t-s)dF(s)$.

Renewal equation

Theorem

If h is bounded then the function

$$H(t) = \int_0^t h(t-s)dU(s)$$

is the unique solution of the renewal equation that is bounded on bounded intervals.

Renewal equation

Proof.

Let $U_n(A) = \sum_{m=0}^n \text{Prob}(T_m \in A)$ and

$$H_n(t) = \int_0^t h(t-s)dU_n(s) = \sum_{m=0}^n (h * F^{*m})(t).$$

Thus $H_{n+1} = h + H_n * F$. Since $U(t) < \infty$, $U_n(t) \uparrow U(t)$. Hence

$$|H(t) - H_n(t)| \leqslant \|h\|_\infty |U(t) - U_n(t)|$$

so $H_n(t) \to H(t)$ uniformly on bounded intervals. Also,

$$|H_n * F(t) - H * F(t)| \leqslant \sup_{s \leqslant t} |H_n(s) - H(s)| \leqslant \|h\|_\infty |U(t) - U_n(t)|.$$

Taking $n \to \infty$, H is a solution of the renewal equation. □

Renewal equation

Proof.

To prove the uniqueness, suppose H_1, H_2 are two solutions, and set $K = H_1 - H_2$ and note $K = K * F$. Iterating gives $K = K * F^{*n} \to 0$ as $n \to \infty$. $\square$

Pedestrian delay

Example

- Consider crossing a road with traffic given by Poisson process with rate λ.
- One unit of time is required to cross the road. Thus the transition time is $\inf\{t : \text{ no arrivals in } (t, t+1]\}$.
- By considering the time of the first arrival, $H(t) = \text{Prob}(M \leqslant t)$ satisfies
$$H(t) = e^{-\lambda} + \int_0^1 H(t-y)\lambda e^{-\lambda y} dy.$$
- Hence, $H(t) = e^{-\lambda} \sum_{n=0}^{\infty} F^{*n}(t)$.