Math 639: Lecture 24

The Gaussian Free Field and Liouville Quantum Gravity

Bob Hough

May 11, 2017

The Gaussian free field

This lecture loosely follows Berestycki's 'Introduction to the Gaussian Free Field and Liouville Quantum Gravity'.

Itô's formula

The multidimensional Itô's formula is as follows.

Theorem (Multidimensional Itô's formula)

Let $\{B(t) : t \geqslant 0\}$ be a d-dimensional Brownian motion and suppose $\{\zeta(s) : s \geqslant 0\}$ is a continuous, adapted stochastic process with values in $\mathbb{R}^m$ and increasing components. Let $f : \mathbb{R}^{d+m} \to \mathbb{R}$ satisfy

- $\partial_i f$ *and* $\partial_{jk} f$, *all* $1 \leqslant j, k \leqslant d$, $d+1 \leqslant i \leqslant d+m$ *are continuous*
- $\mathrm{E} \int_0^t |\nabla_x f(B(s), \zeta(s))|^2 ds < \infty$

then a.s. for all $0 \leqslant s \leqslant t$

$$f(B(s), \zeta(s)) - f(B(0), \zeta(0)) = \int_0^s \nabla_x f(B(u), \zeta(u)) \cdot dB(u)$$
$$+ \int_0^s \nabla_y f(B(u), \zeta(u)) \cdot d\zeta(u) + \frac{1}{2} \int_0^s \Delta_x f(B(u), \zeta(u)) du.$$

Conformal maps

Definition

Let U and V be domains in $\mathbb{R}^2$. A mapping $f : U \to V$ is *conformal* if it is a bijection and preserves angles.

Viewed as a map between domains in $\mathbb{C}$, this is equivalent to f is an analytic bijection.

Conformal invariance of Brownian motion

The following conformal invariance of Brownian motion may be established with Itô's formula.

Theorem

Let U be a domain in the complex plane, $x \in U$, and let $f : U \to V$ be analytic. Let $\{B(t) : t \geqslant 0\}$ be a planar Brownian motion started in x and

$$\tau_U = \inf\{t \geqslant 0 : B(t) \notin U\}$$

its first exit time from the domain U. There exists a planar Brownian motion $\{\tilde{B}(t) : t \geqslant 0\}$ such that, for any $t \in [0, \tau_U)$,

$$f(B(t)) = \tilde{B}(\zeta(t)), \qquad \zeta(t) = \int_0^t |f'(B(s))|^2 ds.$$

If f is conformal, then $\zeta(\tau_U)$ is the first exit time from V by $\{\tilde{B}(t) : t \geqslant 0\}$.

Conformal invariance of Brownian motion

Proof.

- We assume that the domains are bounded.
- Recall that if $f = f_1 + if_2$ then the Cauchy-Riemann equations give ∇f_1 and ∇f_2 are orthogonal, and $|\nabla f_1| = |\nabla f_2| = |f'|$.
- Let $\sigma(t) = \inf\{s \geqslant 0 : \zeta(s) \geqslant t\}$
- Let $\{\tilde{B}(t) : t \geqslant 0\}$ be an independent Brownian motion, and define

$$W(t) = f(B(\sigma(t) \wedge \tau_U)) + \tilde{B}(t) - \tilde{B}(t \wedge \zeta(\tau_U)), \qquad t \geqslant 0.$$

- It suffices to check that $W(t)$ is a Brownian motion. Since it is almost surely continuous, it remains to check the f.d.d.

□

Proof.

- To check the f.d.d. we check for $0 \leqslant s \leqslant t$, and $\lambda \in \mathbb{C}$,

$$\mathrm{E}\left[e^{\langle \lambda, W(t) \rangle} \middle| \mathscr{G}(s)\right] = \exp\left(\frac{1}{2}|\lambda|^2(t-s) + \langle \lambda, W(s) \rangle\right).$$

- It suffices to check

$$\mathrm{E}\left[e^{\langle \lambda, W(t) \rangle} | W(s) = f(x)\right] = \exp\left(\frac{1}{2}|\lambda|^2(t-s) + \langle \lambda, f(x) \rangle\right).$$

We evaluate this at $s = 0$.

□

Conformal invariance of Brownian motion

Proof.

- Calculate

$$\mathrm{E}\left[e^{\langle \lambda, W(t)\rangle}\middle| W(0) = f(x)\right]$$
$$= \mathrm{E}_x \exp\left(\langle \lambda, f(B(\sigma(t)\wedge \tau_U))\rangle + \frac{1}{2}|\lambda|^2 (t - \zeta(\sigma(t)\wedge \tau_U))\right).$$

- We use Itô with

$$F(x,u) = \exp\left(\langle \lambda, f(x)\rangle + \frac{1}{2}|\lambda|^2(t-s)\right).$$

Note $\Delta e^{\langle \lambda, f(x)\rangle} = |\lambda|^2 |f'(x)|^2 e^{\langle \lambda, f(x)\rangle}$.

□

Conformal invariance of Brownian motion

Proof.

- Recall $F(x,u) = \exp\left(\langle \lambda, f(x)\rangle + \frac{1}{2}|\lambda|^2(t-u)\right)$. Set $T = \sigma(t) \wedge \tau_{U_n}$, with $U_n = \{x \in U : d(x, \partial U) \geqslant 1/n\}$.
- Hence

$$F(B(T),\zeta(T)) = F(B(0),\zeta(0)) + \int_0^T \nabla_x F(B(s),\zeta(s)) \cdot dB(s)$$
$$+ \int_0^T \partial_u F(B(s),\zeta(s)) d\zeta(s) + \frac{1}{2}\int_0^T \Delta_x F(B(s),\zeta(s)) ds.$$

- Use $d\zeta(u) = |f'(B(u))|^2 du$ to cancel the two terms on the bottom line. Also, the stochastic integral has mean 0.

□

Conformal invariance of Brownian motion

Proof.

- We thus calculate

$$\mathrm{E}\left[e^{\langle \lambda, W(t)\rangle} \middle| W(0) = f(x)\right] = \mathrm{E}_x\left[F(B(\sigma(t) \wedge \tau_U), \zeta(\sigma(t) \wedge \tau_U))\right]$$

$$= \lim_{n\to\infty} \mathrm{E}_x\left[F(B(T), \zeta(T))\right] = F(x, 0) = \exp\left(\frac{1}{2}|\lambda|^2 t + \langle \lambda, f(x)\rangle\right).$$

□

Discrete case

- Let $G = (V, E)$ be an undirected graph, and let ∂ be a distinguished set of vertices, called the boundary.
- Let $\hat{V} = V \backslash \partial$ be the internal vertices.
- For $x, y \in V$, write $x \sim y$ if x and y are neighbors.
- Let $\{X_n\}$ be a random walk on G, in which at each step the walker chooses uniformly a random neighbor.
- Let P be the transition matrix, $d(x) = \deg(x)$, which is an invariant measure for the walk, and let τ be the first hitting time to the boundary.

Discrete case

Definition (Green function)

The Green function $G(x, y)$ is defined for $x, y \in V$ by putting

$$G(x, y) = \frac{1}{d(y)} \mathrm{E}_x \left(\sum_{n=0}^{\infty} \mathbf{1}_{(X_n = y; \tau > n)} \right).$$

Definition (Discrete Laplacian)

The discrete Laplacian acts on functions on V by

$$\Delta f(x) = \sum_{y \sim x} \frac{1}{d(x)} (f(y) - f(x)).$$

Discrete case

Proposition

The following hold:

1. *Let* $\hat{P}$ *denote the restriction of* P *to* $\hat{V}$*. Then* $(I - \hat{P})^{-1}(x, y) = G(x, y)d(y)$ *for all* $x, y \in \hat{V}$*.*
2. G *is a symmetric nonnegative semidefinite function. That is, one has* $G(x, y) = G(y, x)$ *and if* $(\lambda_x)_{x\in V}$ *is a vector then* $\sum_{x,y\in V} \lambda_x\lambda_y G(x, y) \geqslant 0$*. Equivalently, all eigenvalues are non-negative.*
3. $G(x, \cdot)$ *is discrete harmonic in* $\hat{V}\setminus\{x\}$*, and* $\Delta G(x, \cdot) = -\delta_x(\cdot)$*.*

Discrete case

Definition

The discrete Gaussian free field is the centered Gaussian vector $(h(x))_{x \in V}$ with covariance given by the Green function G.

Note that if $x \in \partial$, then $G(x, y) = 0$ for all $y \in V$ and hence $h(x) = 0$ a.s.

Discrete case

Theorem (Law of the GFF)

The law of $(h(x))_{x\in V}$ *is absolutely continuous with respect to* $dx = \prod_{u\in V} dx_u$*, and the joint pdf is given by*

$$\mathrm{Prob}(h(x) \in A) = \frac{1}{Z}\int_A \exp\left(-\frac{1}{4}\sum_{u\sim v\in V}(y_u - y_v)^2\right)\prod_{u\in V} dy_u.$$

Z *is a normalizing constant, called the partition function.*

The quadratic form appearing in the exponential is called the *Dirichlet energy*.

Discrete case

Proof.

- For a centered Gaussian vector $(Y_1, ..., Y_n)$ with covariance matrix V, the joint density is given by

$$\frac{1}{Z}\exp\left(-\frac{1}{2}y^T V^{-1} y\right)$$

□

Discrete case

Proof.

- Restrict to only variables from $\hat{V}$ and calculate

$$
\begin{aligned}
h(\hat{x})^T G^{-1} h(\hat{x}) &= \sum_{x,y\in\hat{V}} G^{-1}(x,y)h(x)h(y)\\
&= \sum_{x,y\in\hat{V}:x\sim y} -d(x)\hat{P}(x,y)h(x)h(y) + \sum_{x\in\hat{V}} d(x)h(x)^2\\
&= -\sum_{x,y\in V:x\sim y} h(x)h(y) + \sum_{x,y\in V:x\sim y} \frac{1}{2}(h(x)^2+h(y)^2)\\
&= \sum_{x,y\in V:x\sim y} \frac{1}{2}(h(x)-h(y))^2.
\end{aligned}
$$

□

Discrete case

Theorem (Markov property)

Fix $U \subset V$. The discrete GFF $h(x)$ can be decomposed as follows:

$$h = h_0 + \phi$$

where h_0 is a Dirichlet boundary Gaussian free field on U and ϕ is harmonic on U. Moreover, h_0 and ϕ are independent.

We prove a continuum version of this theorem in the next section.

Continuum case

For $D \subset \mathbb{R}^d$ let $p_t^D(x, y)$ be the transition kernel of Brownian motion killed on leaving D. Thus

$$p_t^D(x, y) = p_t(x, y)\pi_t^D(x, y)$$

where

$$p_t(x, y) = \frac{\exp\left(-\frac{|x-y|^2}{2t}\right)}{(2\pi t)^{\frac{d}{2}}}$$

and $\pi_t^D(x, y)$ is the probability that a Brownian bridge of duration t remains in D.

Continuum case

Definition

The Green function $G(x, y) = G_D(x, y)$ is given by

$$G(x, y) = \pi \int_0^\infty p_t^D(x, y) dt.$$

$G(x, x) = \infty$ for all $x \in D$, since $\pi_t^D(x, x) \to 1$ as $t \to 0$. If D is bounded then $G(x, y) < \infty$ for all $x \neq y$.

Continuum case

Example

Suppose $D = \mathbb{H}$ is the upper half plane. Then $p_t^{\mathbb{H}}(x,y) = p_t(x,y) - p_t(x,\overline{y})$ and

$$G_{\mathbb{H}}(x,y) = \log\left|\frac{x-\overline{y}}{x-y}\right|.$$

Continuum case

We now restrict attention to dimension 2.

Proposition

If $T : D \to D'$ is a conformal map (holomorphic and one-to-one), then $G_{T(D)}(T(x), T(y)) = G_D(x, y)$.

Continuum case

Proof.

- Let ϕ be a test function and let $x', y' = T(x), T(y)$.
- Then

$$\int_{D'} G_{D'}(x', y')\phi(y')dy' = \mathrm{E}_{x'}\left[\int_0^{\tau'} \phi(B'_{t'})dt'\right]$$

where B' is Brownian motion and τ' is the first exit time from D'.

- Since $dy' = |T'(y)|^2 dy$

$$\int_{D'} G_{D'}(x', y')\phi(y')dy' = \int_D G_{D'}(T(x), T(y))\phi(T(y))|T'(y)|^2 dy.$$

$\square$

Continuum case

Proof.

- Apply Itô's formula to the RHS, writing $B'_{t'} = T(B_{F^{-1}(t')})$ where $F(t) = \int_0^t |T'(B_s)|^2 ds$ for $t \leqslant \tau$.
- Calculate

$$\mathrm{E}_{x'}\left[\int_0^{\tau'} \phi(B'_{t'})dt'\right] = \mathrm{E}_x\left[\int_0^{\tau} \phi(T(B_s))F'(s)ds\right]$$
$$= \mathrm{E}_x\left[\int_0^{\tau} \phi(T(B_s))|T'(B_s)|^2 ds\right]$$
$$= \int_D G_D(x,y)\phi(T(y))|T'(y)|^2 dy.$$

This proves that $G_{D'}(T(x), T(\cdot)) = G_D(x, \cdot)$ as distributions.

□

Continuum case

Proposition

The following properties hold

1. $G(x, \cdot)$ *is harmonic in* $D \backslash \{x\}$ *and* $\Delta G(x, \cdot) = -2\pi\delta_x(\cdot)$.
2. $G(x, y) = -\log(|x - y|) + \log R(x; D) + o(1)$

where $R(x; D)$ *is the conformal radius of* $x \in D$*, equal to* $|f'(0)|$ *where* f *is any conformal map from the disc* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ *to* D *satisfying* $f(0) = x$.

Continuum case

Proof.

1. Conformal maps preserve harmonicity, which proves the first property.
2. A conformal map from $\mathbb{D} \to \mathbb{D}$ which maps 0 to 0 is of the form $f(z) = e^{i\theta} z$. To check this, let f be such a map, which necessarily maps the boundary to itself, and write $f(z) = \sum_{n=1}^{\infty} a_n z^n$,

$$1 = \frac{1}{2\pi i} \int_{|z|=1} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{|z|=1} f'(z)\overline{f(z)} dz = \sum_{n=1}^{\infty} n|a_n|^2$$

   Since $\sum |a_n|^2 = 1$ we get $|a_1| = 1$ and $a_i = 0$ for $i > 1$. This proves that the conformal radius is well-defined.
3. To calculate the formula for the Green's function, observe that $G_{\mathbb{D}}(0, z) = \log |z|$, as may be checked by using conformal invariance and the map $\phi(z) = \frac{i-z}{i+z}$ from the upper half plane to the disc.

□

Continuum case

In the continuum case, one thinks of the GFF as a "random function" on a domain, with mean 0 and covariance given by the Green function. However, since the Green function is infinite on the diagonal, the GFF is not defined pointwise, and is instead in a negative Sobolev space.

Continuum case

Definition

Let D be a domain on which the Green function is finite off the diagonal. Such a domain is called *Greenian*. Let $\mathscr{M}_+$ be the space of positive measures with compact support on D satisfying

$$\int \rho(dx)\rho(dy)G(x,y) < \infty.$$

Let $\mathscr{M}$ be the collection of signed measures $\rho = \rho_+ - \rho_-$ with $\rho_\pm \in \mathscr{M}_+$. For $\rho_1, \rho_2 \in \mathscr{M}$, define bilinear form

$$\Gamma(\rho_1, \rho_2) := \int_{D^2} G_D(x,y)\rho_1(dx)\rho_2(dy).$$

Define $\Gamma(\rho) = \Gamma(\rho, \rho)$.

Continuum case

Theorem (Zero boundary GFF)

There exists a unique stochastic process $(h_\rho)_{\rho \in \mathscr{M}}$, *indexed by* $\mathscr{M}$, *such that, for every choice* $\rho_1, \ldots, \rho_n$, $(h_{\rho_1}, \ldots, h_{\rho_n})$ *is a centered Gaussian vector with covariance matrix* $\mathrm{Cov}(h_{\rho_i}, h_{\rho_j}) = \Gamma(\rho_i, \rho_j)$.

Definition

The process $(h_\rho)_{\rho \in \mathscr{M}}$ is called the Gaussian free field in D with Dirichlet boundary conditions.

Continuum case

Proof.

- We need to check that the finite-dimensional distributions exist, are uniquely specified, and are consistent. The consistency is immediate, since the f.d.d. are Gaussian vectors.
- To check existence and uniqueness of the f.d.d. we need to show that $\Gamma(\rho_i, \rho_j)$ is symmetric and positive semi-definite.
- Symmetry follows from symmetry of the Greens function, which follows from $p_t^D(x, y) = p_t^D(y, x)$.
- To prove positivity, we need to show,

$$\sum_{i,j} \lambda_i \lambda_j \Gamma(\rho_i, \rho_j) \geqslant 0$$

or, equivalently, $\Gamma(\rho) \geqslant 0$ for all $\rho \in \mathscr{M}$.

□

Continuum case

Proof.

- Recall that Green's formula gives, for f, g in $C^\infty(D)$,

$$\int_D \nabla f \cdot \nabla g = -\int_D f \Delta g + \int_{\partial D} f \frac{\partial g}{\partial n}.$$

- Let $\rho \in C_c^\infty(D)$ and

$$f(x) = -\int G_D(x,y)\rho(y)dy$$

so that $\Delta f = 2\pi\rho$.

□

Continuum case

Proof.

- Hence

$$\begin{aligned}\Gamma(\rho) &= \frac{1}{2\pi}\int_x \rho(x)\int_y G(x,y)\Delta_y f(y)dydx\\ &= \frac{1}{2\pi}\int_x \rho(x)\int_y \Delta_y G(x,y) f(y)dydx\\ &= -\int_x f(x)\Delta f(x)dx = \int_D |\nabla f|^2.\end{aligned}$$

- Thus $\Gamma(\rho,\rho) \geqslant 0$ for $\rho \in C_c^\infty(D)$. The claim holds in general by density.

□

Continuum case

- Going forward we write (h, ρ) for h_ρ. The pairing is linear in ρ as may be checked by noting that the mean and variance of the difference $(h, \alpha\rho + \beta\rho') - \alpha(h, \rho) - \beta(h, \rho')$ are both 0.
- The above description gives the GFF with Dirichlet boundary condition. If f (possibly random) is continuous on the *conformal boundary* of the domain D, then the GFF with boundary condition f is $h = h_0 + \phi$ where ϕ is the harmonic extension of f to D, and where h_0 is an independent solution of the Dirichlet GFF.

Distributions

- Write $\mathscr{D}(D) = C_c^\infty(D)$ for the space of 'test functions' on D. The topology is defined by $f_n \to 0$ in $\mathscr{D}(D)$ if there is a compact set $K \subset D$ such that f_n is supported in K, and f_n and all derivatives converge to 0 uniformly.
- A continuous linear map $u : \mathscr{D}(D) \to \mathbb{R}$ is a *distribution* on D. This space is written $\mathscr{D}'(D)$. It is given the weak-* topology, so that $u_n \to u \in \mathscr{D}'(D)$ if and only if $u_n(\rho) \to u(\rho)$ for all $\rho \in \mathscr{D}(D)$.

Dirichlet energy, Sobolev space

Definition

For $f, g \in \mathscr{D}(D)$, their *Dirichlet inner product* is

$$(f, g)_\nabla := \frac{1}{2\pi} \int_D \nabla f \cdot \nabla g.$$

The Sobolev space $H_0^1(D)$ is the completion of $\mathscr{D}(D)$ with respect to the Dirichlet inner product. This consists of $L^2(D)$ functions whose gradient is also in $L^2(D)$.

The GFF as a random distribution

We now construct the GFF as a random distribution.

- Suppose $h \in \mathscr{D}'(D)$ and $f \in \mathscr{D}(D)$. Set $\rho = -\Delta f$.
- By the definition of distributional derivatives

$$(h, f)_\nabla = -\frac{1}{2\pi}(h, \Delta f) = \frac{1}{2\pi}(h, \rho).$$

- This makes $(h, f)_\nabla$ a centered Gaussian with variance $\frac{\Gamma(\rho)}{(2\pi)^2}$,

$$\mathrm{Var}(h, f)_\nabla = \|f\|_\nabla^2.$$

By polarization, $\mathrm{Cov}((h, f)_\nabla, (h, g)_\nabla) = (f, g)_\nabla$.

The GFF as a random distribution

- Suppose D is a bounded domain, let $\{f_n\}$ be an orthonormal basis of $H_0^1(D)$ which are eigenfunctions of $-\Delta$ with increasing eigenvalue λ_n and let $\{X_n\}$ be a sequence of i.i.d. standard normals. Set
$$h = \sum_n X_n f_n.$$
- Weyl's law gives that $\lambda_n \asymp n$ as $n \to \infty$. Let e_n be f_n scaled to be orthonormal in $L^2(D)$. Since $(f_n, f_n)_\nabla = \frac{-1}{2\pi}(f_n, \Delta f_n)_2 = \frac{\lambda_n}{2\pi}(f_n, f_n)_2$ we have $\sqrt{\frac{2\pi}{\lambda_n}} e_n = f_n$.
- For $s \in \mathbb{R}$, the Sobolev space $H^s(D)$ is
$$H^s(D) = \{f \in \mathscr{D}'(D) : \sum_n (f, e_n)^2 \lambda_n^s < \infty\}$$
with inner product
$$(f, g)_s = \sum_n (f, e_n)(g, e_n)\lambda_n^s.$$

The GFF as a random distribution

- It follows that $h = \sum_n X_n f_n$ converges a.s. in $H^{-\epsilon}(D)$ for every $\epsilon > 0$.
- For all $f \in H_0^1(D)$,

$$(h, f)_\nabla := \sum_n X_n (f_n, f)_\nabla$$

converges in $L^2(\text{Prob})$ and a.s. by the martingale convergence theorem. It's limit is a Gaussian with variance

$$\sum_n (f_n, f)_\nabla^2 = \|f\|_\nabla^2.$$

Markov property

Theorem (Markov property)

Fix $U \subset D$ open and take h a GFF with zero boundary condition on D. Then we may write

$$h = h_0 + \phi$$

where

1. *h_0 is a zero boundary condition GFF on U and is zero outside of U.*
2. *ϕ is harmonic in U.*
3. *h_0 and ϕ are independent.*

Markov property

Proof.

- We first check that $H_0^1(D) = \mathrm{Supp}(U) \oplus \mathrm{Harm}(U)$ where $\mathrm{Supp}(U)$ is the closure of smooth functions of compact support in U and $\mathrm{Harm}(U)$ is functions harmonic in U.
- The orthogonality of $\mathrm{Supp}(U)$ and $\mathrm{Harm}(U)$ follows by the Gauss-Green formula.
- Given $f \in H_0^1(D)$, let f_0 be the orthogonal projection onto $\mathrm{Supp}(U)$, $\phi = f - f_0$.
- For any test function $\psi \in \mathscr{D}(U)$, $(\phi, \psi)_\nabla = 0$, so that

$$\int_D (\Delta\phi)\psi = \int_U (\Delta\phi)\psi = 0.$$

so that $\Delta\phi = 0$ as a distribution in U. By elliptic regularity, ϕ is a smooth function, and harmonic.

□

Markov property

Proof.

- With the L^2 decomposition, let $\{f_n^0\}$ be an o.n. basis of $\mathrm{Supp}(U)$, and $\{\phi_n\}$ an o.n. basis of $\mathrm{Harm}(U)$.
- Let (X_n, Y_n) be an i.i.d. sequence of standard Gaussians, and let $h_0 = \sum_n X_n f_n^0$ and $\phi = \sum_n Y_n \phi_n$.
- We have h_0 converges a.s. in $\mathscr{D}'(D)$ since it is GFF on U.
- Since $h_0 + \phi = h$, converges a.s. in $\mathscr{D}'(D)$, the series defining ϕ converges a.s., hence is a C^∞ harmonic on U.

□

Markov property

Proof.

- With the L^2 decomposition, let $\{f_n^0\}$ be an o.n. basis of $\mathrm{Supp}(U)$, and $\{\phi_n\}$ an o.n. basis of $\mathrm{Harm}(U)$.
- Let (X_n, Y_n) be an i.i.d. sequence of standard Gaussians, and let $h_0 = \sum_n X_n f_n^0$ and $\phi = \sum_n Y_n \phi_n$.
- We have h_0 converges a.s. in $\mathscr{D}'(D)$ since it is GFF on U.
- Since $h_0 + \phi = h$, converges a.s. in $\mathscr{D}'(D)$, the series defining ϕ converges a.s., hence is a C^∞ harmonic on U.

□

Conformal invariance

The Dirichlet inner product is conformally invariant. If $\varphi : D \to D'$ is conformally invariant then

$$\int_{D'} \nabla(f \circ \phi^{-1}) \cdot \nabla(g \circ \phi^{-1}) = \int_D \nabla f \cdot \nabla g.$$

Thus if $\{f_n\}$ is an o.n. basis of $H_0^1(D)$ then $\{f_n \circ \phi^{-1}\}$ is an o.n. basis of $H_0^1(D')$. Thus we obtain the following theorem.

Theorem

If h is a random distribution on $\mathscr{D}'(D)$ with the law of the Gaussian free field on D, then $h \circ \phi^{-1}$ is a GFF on $\mathscr{D}'$.

Circle average

Let D be a bounded domain, let $z \in D$, and let $0 < \epsilon < d(z, \partial D)$. Let $\rho_{z,\epsilon}$ be uniform measure on the circle of radius ϵ centered at z. Notice that

$$\int_{D^2} \rho_{z,\epsilon}(dx)\rho_{z,\epsilon}(dy)G(x,y) < \infty$$

since $\int_0^1 x \log x < \infty$, so that $\rho_{z,\epsilon} \in \mathscr{M}$. Set

$$h_\epsilon(z) = (h, \rho_{z,\epsilon}).$$

Circle average

Theorem

Let h be a GFF on D. Fix $z \in D$ and let $0 < \epsilon_0 < d(z, \partial D)$. For $t \geqslant t_0 = \log(1/\epsilon_0)$, set

$$B_t = h_{e^{-t}}(z).$$

Then $(B_t, t \geqslant t_0)$ has the law of a Brownian motion started from B_{t_0}.

Circle average

Proof.

- Suppose $\epsilon_1 > \epsilon_2$ and we condition on h outside $B(z, \epsilon_1)$. Thus we can write $h = h^0 + \phi$ where ϕ is harmonic on $U = B(z, \epsilon_1)$ and h^0 is GFF in U.
- Then $h_{\epsilon_2}(z) = h^0_{\epsilon_2}(z) + \psi$ where ψ is the circle average of ϕ on the boundary of $B(z, \epsilon_2)$. By harmonicity of ϕ, $\psi = h_{\epsilon_1}(z)$.
- Thus

$$h_{\epsilon_2}(z) = h_{\epsilon_1}(z) + h^0_{\epsilon_2}(z)$$

  which proves the independence of the increments.
- By applying the change of scale, $w \mapsto \frac{w-z}{\epsilon_1}$, which conformally maps the outer circle to the unit circle, we see that the increment $h^0_{\epsilon_2}(z)$ depends only on $r = \frac{\epsilon_2}{\epsilon_1}$. This provides the stationarity.

□

Circle average

Proof.

- To determine the rate of the Brownian motion, it suffices to check that the GFF on $\mathbb{D}$ satisfies $\mathrm{Var}\, h_r(0) = -\log r$ for $0 \leqslant r < 1$.
- We have
$$\mathrm{Var}(h_r(0)) = \int_{\mathbb{D}^2} G_{\mathbb{D}}(x,y)\rho_r(dx)\rho_r(dy).$$
- By the mean value property of $G_{\mathbb{D}}(x,\cdot)$, this is
$$\mathrm{Var}(h_r(0)) = \int_{\mathbb{D}} G_{\mathbb{D}}(x,0)\rho_r(dx) = -\log r.$$

□

Circle average

Theorem

There exists a modification of h such that $(h_\epsilon(z), z \in D, 0 < \epsilon < d(z, \partial D))$ *is jointly Hölder continuous of order* $\gamma < \frac{1}{2}$ *on all compacts of* $(z \in D, 0 < \epsilon < d(z, \partial D))$.

Note: A proof is given in Duplantier and Sheffield, 2011.

Thick points

Definition

Let h be a GFF in D and let $\alpha > 0$. We say a point $z \in D$ is α-thick if

$$\liminf_{\epsilon \to 0} \frac{h_\epsilon(z)}{\log(1/\epsilon)} = \alpha.$$

Thick points

Theorem

Let T_α denote the set of α-thick points. Almost surely

$$\dim T_\alpha = \left(2 - \frac{\alpha^2}{2}\right)_+$$

and T_α is empty if $\alpha \geqslant 2$.

Thick points

Proof sketch.

We sketch only the upper bound. Given $\epsilon > 0$,

$$\begin{aligned}\mathrm{Prob}(h_\epsilon(z) \geqslant \alpha \log(1/\epsilon)) &= \mathrm{Prob}(\eta(0, \log(1/\epsilon) + O(1)) \geqslant \alpha \log(1/\epsilon)) \\ &= \mathrm{Prob}(\eta(0,1) \geqslant \alpha\sqrt{\log(1/\epsilon) + O(1)}) \\ &\leqslant \epsilon^{\alpha^2/2}.\end{aligned}$$

In the square $D = (0,1)^2$, the number of sub-squares of side length ϵ with center z and $h_\epsilon(z)$ satisfying the bound is, on average, $\epsilon^{-2+\alpha^2/2}$. A more elaborate argument bounds the Minkowski dimension by $2 - \frac{\alpha^2}{2}$ a.s. □

Liouville measure

Given $\epsilon > 0$ define $\mu_\epsilon(dz) := e^{\gamma h_\epsilon(z)} \epsilon^{\gamma^2/2} dz$.

Theorem

Suppose $\gamma < 2$. Then the random measure μ_ϵ converges almost surely weakly to a random measure μ, the (bulk) Liouville measure, along the sequence $\epsilon = 2^{-k}$. μ has a.s. no atoms, and for any $A \subset D$ open, we have $\mu(A) > 0$ a.s. In fact,

$$\mathrm{E}\,\mu(A) = \int_A R(z, D)^{\gamma^2/2} dz \in (0, \infty).$$

Liouville measure

Lemma

We have $\operatorname{Var} h_\epsilon(x) = \log(1/\epsilon) + \log R(x, D)$. *In particular,*

$$\mathrm{E}\, \mu_\epsilon(A) = \int_A R(z, D)^{\gamma^2/2} dz.$$

Liouville measure

Proof.

- Calculate

$$\operatorname{Var} h_\epsilon(x) = \Gamma(\rho_{x,\epsilon}) = \int \rho_{x,\epsilon}(dz)\rho_{x,\epsilon}(dw)G(z,w)$$
$$= \int \rho_{x,\epsilon}(dz)G(z,x).$$

- $G(x,\cdot) = -\log|x-\cdot| + \xi(\cdot)$ where ξ is the harmonic extension of $-\log|x-\cdot|$ from the boundary. Thus

$$\operatorname{Var} h_\epsilon(x) = G_\epsilon(x) = \int G(x,y)\rho_{x,\epsilon}(dy) = \log(1/\epsilon) + \xi(x).$$

The conclusion follows since $G(x,y) = \log(1/|x-y|) + \xi(x) + o(1)$ as $y \to x$.

□

Liouville measure

Let S be bounded and open and set $I_\epsilon = \mu_\epsilon(S)$. Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{2}$. We first consider the easier case $\gamma < \sqrt{2}$.

Theorem

We have the estimate $\mathrm{E}((I_\epsilon - I_\delta)^2) \leqslant C\epsilon^{2-\gamma^2}$. *In particular,* I_ϵ *is a Cauchy sequence in* $L^2(\mathrm{Prob})$ *and so converges to a limit in probability. Along* $\epsilon = 2^{-k}$ *this convergence occurs a.s.*

Liouville measure

Proof.

- Let $\overline{h}_\epsilon(z) = \gamma h_\epsilon(z) - (\gamma^2/2)\,\mathrm{Var}(h_\epsilon(z))$ and let $\sigma(dz) = R(z,D)^{\gamma^2/2}$.
- By Fubini

$$\mathrm{E}((I_\epsilon - I_\delta)^2) = \int_{S^2} \mathrm{E}\left[\left(e^{\overline{h}_\epsilon(x)} - e^{\overline{h}_\delta(x)}\right)\left(e^{\overline{h}_\epsilon(y)} - e^{\overline{h}_\delta(y)}\right)\right]\sigma(dx)\sigma(dy).$$

- Write

$$\begin{aligned}&\left(e^{\overline{h}_\epsilon(x)} - e^{\overline{h}_\delta(x)}\right)\left(e^{\overline{h}_\epsilon(y)} - e^{\overline{h}_\delta(y)}\right)\\ &= e^{\overline{h}_\epsilon(x)+\overline{h}_\epsilon(y)}\left(1 - e^{\overline{h}_\delta(x)-\overline{h}_\epsilon(x)}\right)\left(1 - e^{\overline{h}_\delta(y)-\overline{h}_\epsilon(y)}\right).\end{aligned}$$

The three terms now are independent of each other by the Markov property if $|x-y| > 2\epsilon$. In this case the expectation vanishes.

□

Liouville measure

Proof.

- When $|x - y| \leqslant 2\epsilon$,

$$\begin{aligned}
&\mathrm{E}((I_\epsilon - I_\delta)^2) \\
&\leqslant \int_{|x-y|\leqslant 2\epsilon} \sqrt{\mathrm{E}((e^{\overline{h}_\epsilon(x)} - e^{\overline{h}_\delta(x)})^2)\,\mathrm{E}((e^{\overline{h}_\epsilon(y)} - e^{\overline{h}_\delta(y)})^2)}\sigma(dx)\sigma(dy) \\
&\leqslant C \int_{|x-y|\leqslant 2\epsilon} \sqrt{\mathrm{E}(e^{2\overline{h}_\epsilon(x)})\,\mathrm{E}(e^{2\overline{h}_\epsilon(y)})}\sigma(dx)\sigma(dy) \\
&\leqslant C \int_{|x-y|\leqslant 2\epsilon} \epsilon^{\gamma^2} e^{\frac{1}{2}(2\gamma)^2 \log(1/\epsilon)}\sigma(dx)\sigma(dy) \\
&\leqslant C\epsilon^{2-\gamma^2}.
\end{aligned}$$

□

Tilting lemma

Lemma

Let $X = (X_1, ..., X_n)$ *be a Gaussian vector with law* Prob, *with mean* μ *and covariance matrix* V. *Let* $\alpha \in \mathbb{R}^n$ *and define a new probability measure by*

$$\frac{d\,\mathrm{Prob}'}{d\,\mathrm{Prob}} = \frac{e^{\langle \alpha, X\rangle}}{\mathrm{E}[e^{\langle \alpha, X\rangle}]}$$

Under Prob′, X *is a Gaussian vector with covariance matrix* V *and mean* $\mu + V\alpha$.

Tilting lemma

Proof.

It suffices to let $\mu = 0$. The Laplace transform is given by

$$\begin{aligned} \mathrm{E}_{\mathrm{Prob}'}\left[e^{\langle \lambda, X\rangle}\right] &= \frac{\mathrm{E}\left[e^{\langle \lambda+\alpha, X\rangle}\right]}{\mathrm{E}\left[e^{\langle \alpha, X\rangle}\right]} \\ &= \frac{e^{\frac{1}{2}\langle \alpha+\lambda, V(\alpha+\lambda)\rangle}}{e^{\frac{1}{2}\langle \alpha, V\alpha\rangle}} \\ &= e^{\frac{1}{2}\langle \lambda, V\lambda\rangle + \langle \lambda, V\alpha\rangle}. \end{aligned}$$

The term $\langle \lambda, V\lambda\rangle$ corresponds to a Gaussian of variance V. The linear term $\langle \lambda, V\alpha\rangle$ indicates the mean is $V\alpha$. □

Now consider $\gamma \in [\sqrt{2}, 2)$. Let $\alpha > 0$ be fixed. Define 'good' event $G_\epsilon^\alpha(x) = \{h_\epsilon(x) \leqslant \alpha \log(1/\epsilon)\}$.

Lemma (Liouville points are no more than γ-thick)

For $\alpha > \gamma$ we have

$$\mathrm{E}(e^{\overline{h}_\epsilon(x)} \mathbf{1}_{G_\epsilon^\alpha(x)}) \geqslant, 1 - p(\epsilon)$$

where the function p may depend on α and for a fixed $\alpha > \gamma$, $p(\epsilon) \to 0$ as $\epsilon \to 0$ polynomially fast. The same estimate holds if $\overline{h}_\epsilon(x)$ is replaced by $\overline{h}_{\epsilon/2}(x)$.

Liouville measure

Proof.

- Note
$$\mathrm{E}\left[e^{\overline{h}_\epsilon(x)}\mathbf{1}_{G_\epsilon^\alpha(x)}\right] = \mathrm{Prob}'(G_\epsilon^\alpha(x)), \qquad \frac{d\,\mathrm{Prob}'}{d\,\mathrm{Prob}}(x) = e^{\overline{h}_\epsilon(x)}.$$
- By the tilting lemma, under Prob′ the process $X_s = h_{e^{-s}}(x)$ has the same covariance and its mean is $\gamma\,\mathrm{Cov}(X_s, X_t)$.
- Thus, under Prob′, X_s is Brownian motion with drift γ. It follows that the probability that $X_t \geqslant \alpha t$ is exponentially small in t for t large, or polynomially small in ϵ.
- Changing ϵ to $\epsilon/2$ shifts t by $\log 2$. The conclusion is the same.

□

Liouville measure

Fix $\alpha > \gamma$ and introduce

$$J_\epsilon = \int_S e^{\overline{h}_\epsilon(x)} \mathbf{1}_{G_\epsilon(x)} \sigma(dx), \qquad J'_{\epsilon/2} = \int_S e^{\overline{h}_{\epsilon/2}(x)} \mathbf{1}_{G_\epsilon(x)} \sigma(dx)$$

with $G_\epsilon(x) = G_\epsilon^\alpha(x)$. By the previous lemma, $\mathrm{E}(|I_\epsilon - J_\epsilon|) \leqslant p(\epsilon)|S|$ and $\mathrm{E}(|I_{\epsilon/2} - J'_{\epsilon/2}|) \leqslant p(\epsilon)|S|$ also tends to zero.

Liouville measure

Lemma

We have $\mathrm{E}((J_\epsilon - J'_{\epsilon/2})^2) \leqslant \epsilon^r$ *for some* $r > 0$*. In particular,* I_ϵ *is a Cauchy sequence in* L^1 *and so converges to a limit in probability. Along* $\epsilon = 2^{-k}$*, this convergence occurs almost surely.*

Proof.

- Observe that if $|x-y| \geqslant 2\epsilon$ then the increments $h_\epsilon(x) - h_{\epsilon/2}(x)$ and $h_\epsilon(y) - h_{\epsilon/2}(y)$ are independent of each other and also the σ-algebra generated by h outside the balls of radius ϵ about x and y, in particular of $G_\epsilon(x)$ and $G_\epsilon(y)$.
- By Cauchy-Schwarz,

$$\begin{aligned} &\mathrm{E}\left((J_\epsilon - J'_{\epsilon/2})^2\right) \\ &\leqslant C \int_{|x-y|\leqslant 2\epsilon} \sqrt{\mathrm{E}(e^{2\overline{h}_\epsilon(x)}\mathbf{1}_{G_\epsilon(x)})\,\mathrm{E}(e^{2\overline{h}_\epsilon(y)}\mathbf{1}_{G_\epsilon(y)})}\sigma(dx)\sigma(dy). \end{aligned}$$

□

Liouville measure

Proof.

- Observe

$$\mathrm{E}(e^{2\overline{h}_\epsilon(x)}\mathbf{1}_{G_\epsilon(x)}) \leqslant O(1)\epsilon^{-\gamma^2}\mathbb{Q}(h_\epsilon(x) \leqslant \alpha \log 1/\epsilon)$$

where $\mathbb{Q}$ has density $\frac{e^{2\overline{h}_\epsilon(x)}}{\mathrm{E}[e^{2\overline{h}_\epsilon(x)}]}$.

- By the tilting lemma $h_\epsilon(x)$ is a normal random variable with mean $2\gamma \log(1/\epsilon) + O(1)$ and variance $\log 1/\epsilon + O(1)$. Thus

$$\mathbb{Q}(h_\epsilon(x) \leqslant \alpha \log 1/\epsilon) \leqslant O(1) \exp\left(-\frac{(2\gamma - \alpha)^2}{2} \log 1/\epsilon\right).$$

Thus $\mathrm{E}\left((J_\epsilon - J'_{\epsilon/2})^2\right) \leqslant O(1)\epsilon^{2-\gamma^2}\epsilon^{\frac{1}{2}(2\gamma-\alpha)^2}$. Choosing α suff. close to γ makes this $< \epsilon^r$ for some $r > 0$.

□