Math 639: Lecture 23

Stochastic integrals and applications

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Stochastic integrals

This lecture follows Mörters and Peres, Chapter 7.

Downcrossings

Definition

Given $a < b$ define a sequence of stopping times $\tau_0 = 0$ and, for $j \geqslant 1$,

$$\sigma_j = \inf\{t > \tau_{j-1} : B(t) = b\}, \qquad \tau_j = \inf\{t > \sigma_j : B(t) = a\}.$$

We call the random function

$$B^{(j)} : [0, \tau_j - \sigma_j] \to \mathbb{R}, \qquad B^{(j)}(s) = B(\sigma_j + s)$$

the *jth downcrossing* of $[a, b]$. For every $t > 0$, denote

$$D(a, b, t) = \max\{j \in \mathbb{N} : \tau_j \leqslant t\}$$

the number of downcrossings of the interval $[a, b]$ before time t.

Local time at 0

Theorem

There exists a stochastic process $\{L(t) : t \geqslant 0\}$ called the local time at zero such that for all sequences $a_n \uparrow 0$ and $b_n \downarrow 0$ with $a_n < b_n$, a.s.

$$\lim_{n\to\infty} 2(b_n - a_n)D(a_n, b_n, t) = L(t) \qquad \forall\ t > 0.$$

Moreover, this process is almost surely locally α-Hölder continuous for any $\alpha < 1/2$.

Lévy's theorem

Theorem (Lévy)

The local time at zero $\{L(t) : t \geqslant 0\}$ *and the maximum process* $\{M(t) : t \geqslant 0\}$ *of a standard Brownian motion have the same distribution.*

Occupation measure

Theorem

For all sequences $a_n \uparrow 0$ and $b_n \downarrow 0$ with $a_n < b_n$, a.s.

$$\lim_{n\to\infty} \frac{1}{b_n - a_n} \int_0^t \mathbf{1}(a_n \leqslant B(s) \leqslant b_n) ds = L(t), \qquad \forall t > 0.$$

Occupation measure

Theorem

For linear Brownian motion $\{B(t) : t \geqslant 0\}$*, almost surely, for any bounded measurable* $g : \mathbb{R} \to \mathbb{R}$ *and* $t > 0$,

$$\int g(a) d\mu_t(a) = \int_0^t g(B(s)) ds = \int_{-\infty}^{\infty} g(a) L^a(t) da.$$

Trotter's theorem

Given $a \in \mathbb{R}$ and integer n, let $I(a,n) = [j(a)2^{-n}, (j(a)+1)2^{-n})$ be the unique dyadic interval containing a. For a standard Brownian motion $\{B(t) : t \geqslant 0\}$ denote by $D^{(n)}(a,t)$ the number of downcrossings of the interval $I(a,n)$ before time t.

Theorem (Trotter's theorem)

Let $\{B(t) : t \geqslant 0\}$ be a standard linear Brownian motion and let $D^{(n)}(a,t)$ be the number of downcrossings before time t of the nth stage dyadic interval containing a. Then, a.s.

$$L^a(t) = \lim_{n\to\infty} 2^{-n+1} D^{(n)}(a,t),$$

exists for all $a \in \mathbb{R}$ and $t \geqslant 0$. Moreover, for every $\gamma < \frac{1}{2}$, the random field $\{L^a(t) : a \in \mathbb{R}, t \geqslant 0\}$ is a.s. locally γ-Hölder continuous.

Ray-Knight Theorem

Theorem (Ray-Knight Theorem)

Suppose $a > 0$ *and* $\{B(t) : 0 \leqslant t \leqslant T\}$ *is a linear Brownian motion started at* a *and stopped at time* $T = \inf\{t \geqslant 0 : B(t) = 0\}$, *when it reaches level zero for the first time. Then*

$$\{L^x(T) : 0 \leqslant x \leqslant a\} \overset{d}{=} \{|W(x)|^2 : 0 \leqslant x \leqslant a\},$$

where $\{W(x) : x \geqslant 0\}$ *is a standard planar Brownian motion.*

Stochastic integrals

Since the Brownian motion a.s. has unbounded variation it is not possible to define integrals $\int_0^t f(s)dB(s)$ by Lebesgue-Stieltjes integration. Thus stochastic integration is needed.

Stochastic integrals

Definition

Assume the filtration $(\mathscr{F}(t) : t \geqslant 0)$ is *complete* in the sense that it contains all null sets of $\mathscr{A}$. A process $\{X(t, \omega) : t \geqslant 0, \omega \in \Omega\}$ is called *progressively measurable* if for each $t \geqslant 0$ the mapping $X : [0, t] \times \Omega \to \mathbb{R}$ is measurable w.r.t. the σ-algebra $\mathscr{B}([0, t]) \otimes \mathscr{F}(t)$.

Stochastic integrals

Lemma

Any process $\{X(t) : t \geqslant 0\}$ *which is adapted and either right or left continuous is also progressively measurable.*

Stochastic integrals

Proof.

- Assume that $\{X(t) : t \geqslant 0\}$ is right-continuous. Let $t > 0$ and, for a positive integer n and $0 \leqslant s \leqslant t$ set $X_n(0, \omega) = X(0, \omega)$

$$X_n(s, \omega) = X\left(\frac{(k+1)t}{2^n}, \omega\right), \qquad kt2^{-n} < s \leqslant (k+1)t2^{-n}.$$

- $(s, \omega) \mapsto X_n(s, \omega)$ is $\mathscr{B}([0, t]) \otimes \mathscr{F}(t)$ measurable. By right-continuity we have $\lim_{n\uparrow\infty} X_n(s, \omega) = X(s, \omega)$ for all $s \in [0, t]$ and $\omega \in \Omega$.
- Thus the limit map $(s, \omega) \mapsto X(s, \omega)$ is also $\mathscr{B}([0, t]) \otimes \mathscr{F}(t)$.
- The claim in case of left continuity is similar.

$\square$

Stochastic integrals

A *progressively measurable step function* $\{H(t,\omega) : t \geqslant 0, \omega \in \Omega\}$ is a function of the form

$$H(t,\omega) = \sum_{i=1}^{k} A_i(\omega)\mathbf{1}_{(t_i,t_{i+1}]}, \qquad 0 \leqslant t_1 \leqslant \ldots \leqslant t_{k+1}$$

and $\mathscr{F}(t_i)$-measurable A_i. For such functions, define

$$\int_0^\infty H(s)dB(s) := \sum_{i=1}^{k} A_i(B(t_{i+1}) - B(t_i)).$$

For a progressively measurable process H, define

$$\|H\|_2^2 := \mathrm{E}\int_0^\infty H(s)^2 ds.$$

Lemma

For every progressively measurable process $\{H(s,\omega) : s \geqslant 0, \omega \in \Omega\}$ satisfying $\mathrm{E}\int_0^\infty H(s)^2 ds < \infty$ there exists a sequence $\{H_n : n \in \mathbb{N}\}$ of progressively measurable step processes such that $\lim_{n\to\infty} \|H_n - H\|_2 = 0$.

Stochastic integrals

Proof.

- First truncate $H(s,\omega)$ by setting $H_n(s,\omega) = 0$ for $s > n$, $H_n(s,\omega) = H(s,\omega)$ for $s \leqslant n$.
- Next replace $H_n(s,\omega) = H(s,\omega) \wedge n$.
- Next replace $H_n(s,\omega) = n\int_{s-1/n}^{s} H(t,\omega)dt$, which makes H continuous.
- Finally set $H_n(s,\omega) = H(j/n,\omega)$ for $j/n \leqslant s < (j+1)/n$.

$\square$

Lemma

Let H be a progressively measurable step process and $\mathrm{E}\int_0^\infty H(s)^2 ds < \infty$, *then*

$$\mathrm{E}\left[\left(\int_0^\infty H(s)dB(s)\right)^2\right] = \mathrm{E}\int_0^\infty H(s)^2 ds.$$

Stochastic integrals

Proof.

Write $H = \sum_{i=1}^k A_i \mathbf{1}_{(a_i, a_{i+1}]}$ and expand the square

$$
\begin{aligned}
&\mathrm{E}\left[\left(\int_0^\infty H(s)dB(s)\right)^2\right] \\
&= \mathrm{E}\left[\sum_{i,j=1}^k A_i A_j (B(a_{i+1}) - B(a_i))(B(a_{j+1}) - B(a_j))\right] \\
&= 2\sum_{i=1}^k \sum_{j=i+1}^k \mathrm{E}\left[A_i A_j (B(a_{i+1}) - B(a_i))\, \mathrm{E}\left[B(a_{j+1}) - B(a_j) \middle| \mathscr{F}(a_j)\right]\right] \\
&\qquad + \sum_{i=1}^k \mathrm{E}\left[A_i^2 (B(a_{i+1}) - B(a_i))^2\right]
\end{aligned}
$$

$\square$

Stochastic integrals

Proof.

Only the diagonal terms survive, leaving

$$\sum_{i=1}^{k} \mathrm{E}\left[A_i^2\right](a_{i+1} - a_i) = \mathrm{E}\int_0^\infty H(s)^2 ds.$$

$\square$

Stochastic integrals

Theorem

Suppose $\{H_n : n \in \mathbb{N}\}$ is a sequence of progressively measurable step processes and H a progressively measurable process such that

$$\lim_{n\to\infty} \mathrm{E} \int_0^\infty (H_n(s) - H(s))^2 ds = 0,$$

then

$$\lim_{n\to\infty} \int_0^\infty H_n(s) dB(s) =: \int_0^\infty H(s) dB(s)$$

exists as a limit in the L^2-sense and is independent of the choice of $\{H_n : n \in \mathbb{N}\}$. Moreover, we have

$$\mathrm{E}\left[\left(\int_0^\infty H(s) dB(s)\right)^2\right] = \mathrm{E} \int_0^\infty H(s)^2 ds.$$

Stochastic integrals

Proof.

By the previous lemma, the sequence of step functions have integrals that are Cauchy in L^2, hence converge there. The last statement is the convergence of L^2 norms. $\square$

Stochastic integrals

If

$$\sum_{n=1}^{\infty} \mathrm{E} \int_0^{\infty} (H_n(s) - H(s))^2 ds < \infty,$$

then a.s.

$$\sum_{n=1}^{\infty} \left[\int_0^{\infty} H_n(s) dB(s) - \int_0^{\infty} H(s) dB(s) \right]^2 < \infty.$$

which implies $\lim_{n\to\infty} \int_0^{\infty} H_n(s) dB(s) = \int_0^{\infty} H(s) dB(s)$ a.s.

Stochastic integrals

Definition

Suppose $\{H(s,\omega) : s \geqslant 0, \omega \in \Omega\}$ is progressively measurable with $\mathrm{E}\int_0^\infty H(s,\omega)^2 ds < \infty$. Define the progressively measurable process $\{H^t(s,\omega) : s \geqslant 0, \omega \in \Omega\}$ by

$$H^t(s,\omega) = H(s,\omega)\mathbf{1}(s \leqslant t).$$

The *stochastic integral up to t* is defined as,

$$\int_0^t H(s)dB(s) := \int_0^\infty H^t(s)dB(s).$$

Definition

We say that a stochastic process $\{X(t) : t \geqslant 0\}$ is a *modification* of a process $\{Y(t) : t \geqslant 0\}$ if, for every $t \geqslant 0$, we have

$$\mathrm{Prob}(X(t) = Y(t)) = 1.$$

Stochastic integrals

Theorem

Suppose the process $\{H(s,\omega) : s \geqslant 0, \omega \in \Omega\}$ *is progressively measurable with*

$$\mathrm{E}\int_0^t H(s,\omega)^2 ds < \infty, \qquad t \geqslant 0.$$

Then there exists an almost surely continuous modification of $\{\int_0^t H(s)dB(s) : t \geqslant 0\}$*. Moreover, this process is a martingale and hence*

$$\mathrm{E}\int_0^t H(s)dB(s) = 0, \qquad t \geqslant 0.$$

Stochastic integrals

Proof.

- Let t_0 be a large integer and let H_n be a sequence of step processes with $\|H_n - H^{t_0}\|_2 \to 0$. Then
$$\mathrm{E}\left[\left(\int_0^\infty (H_n(s) - H^{t_0}(s))dB(s)\right)^2\right] \to 0.$$
- Since $\int_0^s H_n(u)dB(u)$ is $\mathscr{F}(s)$-measurable and $\int_s^t H_n(u)dB(u)$ is independent of $\mathscr{F}(s)$,
$$\left\{\int_0^t H_n(u)dB(u) : 0 \leqslant t \leqslant t_0\right\}$$
is a martingale.

□

Stochastic integrals

Proof.

- Define
$$X(t) = \mathrm{E}\left[\int_0^{t_0} H(s)dB(s)\middle|\mathscr{F}(t)\right],$$
so that $\{X(t) : 0 \leqslant t \leqslant t_0\}$ is also a martingale.
- By Doob's maximal inequality,
$$\mathrm{E}\left[\sup_{0\leqslant t\leqslant t_0}\left(\int_0^t H_n(s)dB(s) - X(t)\right)^2\right] \leqslant 4\,\mathrm{E}\left[\left(\int_0^{t_0}(H_n(s) - H(s))dB(s)\right)^2\right].$$
- This exhibits $X(t)$ as the uniform limit of continuous processes, as wanted.

$\square$

Stochastic integrals

Theorem

Suppose $f : \mathbb{R} \to \mathbb{R}$ *is continuous,* $t > 0$ *and* $0 = t_1^{(n)} < \ldots < t_n^{(n)} = t$ *are partitions of the interval* $[0, t]$*, such that the mesh converges to 0. Then, in probability,*

$$\sum_{j=1}^{n-1} f(B(t_j^{(n)})) \left(B(t_{j+1}^{(n)}) - B(t_j^{(n)})\right)^2 \to \int_0^t f(B(s))ds.$$

Stochastic integrals

Proof.

- Let T be the first exit time from a compact interval. It suffices to prove the statement for Brownian motion stopped at T, as the interval may be chosen to make $\mathrm{Prob}(T < t)$ arbitarily small.
- By continuity of f, a.s.

$$\lim_{n\to\infty} \sum_{j=1}^{n-1} f(B(t_j^{(n)} \wedge T)) \left(t_{j+1}^{(n)} \wedge T - t_j^{(n)} \wedge T\right) = \int_0^{t\wedge T} f(B(s))ds.$$

$\square$

Stochastic integrals

Proof.

Since $\{B(t)^2 - t : t \geqslant 0\}$ is a martingale, for all $r \leqslant s$,

$$\mathrm{E}\left[(B(s) - B(r))^2 - (s - r)|\mathscr{F}(r)\right] = 0,$$

$$\mathrm{E}\Bigg[\Bigg(\sum_{j=1}^{n-1} f(B(t_j \wedge T))$$

$$\left((B(t_{j+1} \wedge T) - B(t_j \wedge T))^2 - (t_{j+1} \wedge T - t_j \wedge T)\right)\Bigg)^2\Bigg]$$

$$= \sum_{j=1}^{n-1} \mathrm{E}\Bigg[f(B(t_j \wedge T))^2$$

$$\left((B(t_{j+1} \wedge T) - B(t_j \wedge T))^2 - (t_{j+1} \wedge T - t_j \wedge T)\right)^2\Bigg]$$

Proof.

Bound f in sup norm by a constant and bound the remaining part of the sum by

$$\sum_{j=1}^{n-1} \mathrm{E}\left[(B(t_{j+1} \wedge T) - B(t_j \wedge T))^4\right] + \sum_{j=1}^{n-1} \mathrm{E}\left[(t_{j+1} \wedge T - t_j \wedge T)^2\right],$$

which, by Brownian scaling, is bounded by a constant times

$$\sum_{j=1}^{n-1} (t_{j+1} - t_j)^2 \leqslant t\Delta(n)$$

which tends to zero as the mesh does. $\square$

Itô's formula I

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable such that $\mathrm{E} \int_0^t f'(B(s))^2 ds < \infty$ for some $t > 0$. Then, almost surely, for all $0 \leqslant s \leqslant t$,

$$f(B(s)) - f(B(0)) = \int_0^s f'(B(u)) dB(u) + \frac{1}{2} \int_0^s f''(B(u)) du.$$

Itô's formula I

Proof.

- Denote the modulus of continuity of f'' on $[-M, M]$ by

$$\omega(\delta, M) := \sup_{\substack{x,y\in[-M,M]\\ |x-y|<\delta}} |f''(x) - f''(y)|.$$

By Taylor's formula, for any $x, y \in [-M, M]$ with $|x - y| < \delta$,

$$\left| f(y) - f(x) - f'(x)(y-x) - \frac{1}{2} f''(x)(y-x)^2 \right| \leqslant \omega(\delta, M)(y-x)^2.$$

$\square$

Itô's formula I

Proof.

- For any sequence $0 = t_1 < \ldots < t_n = t$ with $\delta_B := \max_{1 \leqslant i \leqslant n-1} |B(t_{i+1}) - B(t_i)|$ and $M_B = \max_{0 \leqslant s \leqslant t} |B(s)|$,

$$\left| \sum_{i=1}^{n-1} (f(B(t_{i+1})) - f(B(t_i))) - \sum_{i=1}^{n-1} f'(B(t_i))(B(t_{i+1}) - B(t_i)) \right.$$
$$\left. - \sum_{i=1}^{n-1} \frac{1}{2} f''(B(t_i))(B(t_{i+1}) - B(t_i))^2 \right|$$
$$\leqslant \omega(\delta_B, M_B) \sum_{i=1}^{n-1} (B(t_{i+1}) - B(t_i))^2.$$

□

Itô's formula I

Proof.

- Choosing a sequence of partitions with mesh size going to 0, the sums converge to integrals on the left, and the sum on the right converges to t a.s., while ω converges to 0.
- This gives the formula at rational t, and everywhere by continuity.

$\square$

Itô's formula II

Theorem

Suppose $\{\zeta(s) : s \geqslant 0\}$ is an increasing, continuous adapted stochastic process. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable in the x-coordinate, and once continuously differentiable in the y-coordinate. Assume that

$$\mathrm{E} \int_0^t \left[\partial_x f(B(s), \zeta(s))\right]^2 ds < \infty,$$

for some $t > 0$. Then, a.s. for all $0 \leqslant s \leqslant t$,

$$\begin{aligned} f(B(s), \zeta(s)) - f(B(0), \zeta(0)) &= \int_0^s \partial_x f(B(u), \zeta(u)) dB(u) \\ &+ \int_0^s \partial_y f(B(u), \zeta(u)) d\zeta(u) + \frac{1}{2} \int_0^s \partial_{xx} f(B(u), \zeta(u)) du. \end{aligned}$$

There is also a multi-dimensional version, see MP pp. 197–200.

Tanaka's formula

Theorem (Tanaka's formula)

Let $\{B(t) : t \geqslant 0\}$ *be linear Brownian motion. Then, for every* $a \in \mathbb{R}$*, almost surely, for all* $t > 0$*,*

$$|B(t) - a| - |B(0) - a| = \int_0^t \operatorname{sgn}(B(s) - a)dB(s) + L^a(t).$$

Tanaka's formula

Corollary

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable such that f' has compact support, but do not assume that f'' is continuous. Then

$$f(B(t)) - f(B(0)) = \int_0^t f'(B(s))dB(s) + \frac{1}{2}\int_0^t f''(B(s))ds.$$

Tanaka's formula

Proof.

Write

$$f'(x) = \frac{1}{2}\int \operatorname{sgn}(x-a)f''(a)da + c, \quad f(x) = \frac{1}{2}\int |x-a|f''(a)da + cx + b.$$

Multiply Tanaka's formula by $\frac{1}{2}f''(a)da$ and integrate to obtain

$$f(B(t)) - f(B(0)) = \int_0^t f'(B(s))dB(s) + \frac{1}{2}\int L^a(t)f''(a)da.$$

$\square$

Tanaka's formula

Define

$$\tilde{L}^a(t) := |B(t) - a| - |B(0) - a| - \int_0^t \operatorname{sgn}(B(s) - a)dB(s).$$

Lemma

For every $t \geqslant 0$ and $a \in \mathbb{R}$,

$$\tilde{L}^a(t) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}_{(a,a+\epsilon)}(B(s))ds$$

in probability.

Tanaka's formula

Proof.

- Using the strong Markov property, reduce to the case $a = 0$.
- Note that, for any $\delta > 0$ we can find smooth functions $g, h : \mathbb{R} \to [0, 1]$ with compact support such that $g \leqslant 1_{(0,1)} \leqslant h$ and $\int g = 1 - \delta$, $\int h = 1 + \delta$.
- Let $f : \mathbb{R} \to [0, 1]$ smooth, compactly supported in $[-1, 2]$, $\int f = 1$, and let

$$f_\epsilon(x) = \epsilon^{-1} \int |x - a| f(\epsilon^{-1} a) da = \int |x - \epsilon a| f(a) da.$$

$$f_\epsilon'(x) = \int \operatorname{sgn}(x - \epsilon a) f(a) da$$
$$f_\epsilon''(x) = 2\epsilon^{-1} f(\epsilon^{-1} x).$$

$\square$

Tanaka's formula

Proof.

- Itô's formula gives

$$f_\epsilon(B(t)) - f_\epsilon(B(0)) - \int_0^t f_\epsilon'(B(s))dB(s) = \epsilon^{-1}\int_0^t f(\epsilon^{-1}B(s))ds.$$

- Since $f_\epsilon(x) \to |x|$ uniformly, we have

$$f_\epsilon(B(t)) - f_\epsilon(B(0)) \to |B(t)| - |B(0)|$$

in probability as $\epsilon \to 0$.

□

Tanaka's formula

Proof.

- By the isometry property,

$$\mathrm{E}\left[\left(\int_0^t \mathrm{sgn}(B(s))dB(s) - \int_0^t f_\epsilon'(B(s))dB(s)\right)^2\right]$$
$$= \mathrm{E}\int_0^t (\mathrm{sgn}(B(s)) - f_\epsilon'(B(s)))^2 ds.$$

  This converges to 0 as $\epsilon \downarrow 0$ by bounded convergence.
- Meanwhile $\epsilon^{-1}\int_0^t f(\epsilon^{-1}B(s))ds \to \tilde{L}^0(t)$.

□

Tanaka's formula

Proof of Tanaka's formula.

- Fix $t \geqslant 0$ and recall that a.s. the occupation measure μ_t given by $\mu_t(A) = \int_0^t \mathbf{1}_A(B(s))ds$ has a continuous density given by $\{L^a(t) : a \in \mathbb{R}\}$.
- Thus, for every $a \in \mathbb{R}$,

$$L^a(t) = \lim_{\epsilon \downarrow 0} \frac{\mu_t(a, a+\epsilon)}{\epsilon} = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}_{(a,a+\epsilon)}(B(s))ds.$$

- By the previous lemma, for every $a \in \mathbb{R}$ and $t \geqslant 0$, $L^a(t) = \tilde{L}^a(t)$ a.s.
- Since, for any $a \in \mathbb{R}$, $\{L^a(t) : t \geqslant 0\}$ and $\{\tilde{L}^a(t) : t \geqslant 0\}$ are almost surely continuous, so that they agree.

□

Lévy's theorem

Theorem (Lévy)

The processes

$$\{(|B(t)|, L^0(t)) : t \geqslant 0\}, \qquad \{(M(t) - B(t), M(t)) : t \geqslant 0\}$$

have the same distribution.

Lemma

For every $a \in \mathbb{R}$, the process $\{W(t) : t \geqslant 0\}$ given by

$$W(t) = \int_0^t \operatorname{sgn}(B(s) - a)\, dB(s).$$

Lévy's theorem

Proof.

- Suppose without loss that $a < 0$.
- Let $T = \inf\{t > 0 : B(t) = a\}$ so that $W(t) = B(t)$ for all $t \leqslant T$.
- $\{\tilde{B}(t) : t \geqslant 0\}$ defined by $\tilde{B}(t) = B(t+T) - a$ is independent of $\{W(t) : 0 \leqslant t \leqslant T\}$. We have

$$W(t+T) - W(T) = \int_0^t \operatorname{sgn}(\tilde{B}(s)) d\tilde{B}(s),$$

so now assume $a = 0$.

$\square$

Lévy's theorem

Proof.

- Choose $s = t_1^{(n)} < \cdots < t_n^{(n)} = t$ with mesh $\Delta(n) \downarrow 0$ and approximate the progressively measurable process $\mathrm{sgn}(B(u))$ by the step processes

$$H_n(u) = \mathrm{sgn}(B(t_j^{(n)})), \qquad t_j^{(n)} < u \leqslant t_{j+1}^{(n)}.$$

- Since the zero set of Brownian motion is a closed set of measure 0, $\lim \mathrm{E} \int_s^t (H_n(u) - H(u))^2 du = 0$.
- It follows that $W(t) - W(s)$ is the L^2-limit

$$\lim_{n\to\infty} \int_s^t H_n(u) dB(u) = \lim \sum_{j=1}^{n-1} \mathrm{sgn}\left(B(t_j^{(n)})\right)\left(B(t_{j+1}^{(n)}) - B(t_j^{(n)})\right).$$

□

Lévy's theorem

Proof.

- Each term in the limit is a mean zero Gaussian of variance $t - s$, so the limit is also.

$\square$

Lévy's theorem

Proof of Lévy's theorem.

- By Tanaka's formula,

$$|B(t)| = \int_0^t \mathrm{sgn}(B(s))dB(s) + L^0(t) = W(t) + L^0(t).$$

- Let $\tilde{W}(t) = -W(t)$ and $\tilde{M}(t)$ be the associated maximal process.
- We claim that $\tilde{M}(t) = L^0(t)$, which suffices, since then

$$\{(|B(t)|, L^0(t)) : t \geqslant 0\}, \qquad \{(\tilde{M}(t) - \tilde{W}(t), \tilde{M}(t)) : t \geqslant 0\}.$$

  agree pointwise.
- To check the equality, first note that $\tilde{W}(s) = L^0(s) - |B(s)| \leqslant L^0(s)$, so that $\tilde{M}(t) \leqslant L^0(t)$. On the other hand, $L^0(t)$ increases only on $\{t : B(t) = 0\}$ where we have $L^0(t) = \tilde{W}(t) \leqslant \tilde{M}(t)$.

□

Heat equation

Definition

Let $U \subset \mathbb{R}^d$ be either open and bounded, or $U = \mathbb{R}^d$. A twice differentiable function $u : (0, \infty) \times U \to [0, \infty)$ is said to solve the *heat equation with heat dissipation rate* $V : U \to \mathbb{R}$ and initial condition $f : U \to [0, \infty)$ on U if we have

- $\lim_{x \to x_0, t \downarrow 0} u(t, x) = f(x_0)$, whenever $x_0 \in U$
- $\lim_{x \to x_0, t \to t_0} u(t, x) = 0$, whenever $x_0 \in \partial U$
- $\partial_t u(t, x) = \frac{1}{2} \Delta_x u(t, x) + V(x) u(t, x)$ on $(0, \infty) \times U$.

Here Δ_x is the Laplacian, acting on the space variables x.

This formula describes the temperature $u(t, x)$ at time t and location x, subject to heating rate V and with 0 boundary condition.

Heat equation

Theorem

Suppose $V : \mathbb{R}^d \to \mathbb{R}$ is bounded. Then $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$u(t,x) = \mathrm{E}_x \left\{ \exp\left(\int_0^t V(B(r))dr \right) \right\},$$

solves the heat equation on $\mathbb{R}^d$ with dissipative rate V and initial condition one.

Heat equation

Proof.

- We check this by Taylor expansion.
- Let $a_0(x,t) := 1$ and, for $n \geqslant 1$,

$$a_n(x,t) := \frac{1}{n!} \mathrm{E}_x \left[\int_0^t \cdots \int_0^t V(B(t_1)) \cdots V(B(t_n)) dt_1 \dots dt_n \right]$$
$$= \mathrm{E}_x \left[\int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{n-1}}^t dt_n V(B(t_1)) \cdots V(B(t_n)) \right]$$
$$= \int dx_1 \cdots \int dx_n \int_0^t dt_1 \cdots \int_{t_{n-1}}^t dt_n \prod_{i=1}^n V(x_i) \prod_{i=1}^n p(t_i - t_{i-1}, x_{i-1}, x_i)$$

with $x_0 = x$ and $t_0 = 0$.

$\square$

Heat equation

Proof.

- Using $\frac{1}{2}\Delta_x p(t_1, x, x_1) = \partial_{t_1} p(t_1, x, x_1)$ and integrating by parts

$$\begin{aligned}\frac{1}{2}\Delta_x a_n(x,t) &= \int dx_1 V(x_1) \int_0^t dt_1 \partial_{t_1} p(t_1, x, x_1) a_{n-1}(x_1, t - t_1)\\ &= -\int dx_1 V(x_1) \int_0^t dt_1 p(t_1, x, x_1) \partial_{t_1} a_{n-1}(x, t - t_1)\\ &\quad - V(x) a_{n-1}(x,t)\\ &= \partial_t a_n(x,t) - V(x) a_{n-1}(x,t).\end{aligned}$$

- Adding terms justifies solution of the differential equation.

$\square$

Heat equation

Theorem

If u is a bounded, twice continuously differentiable solution of the heat equation on the domain U, with zero dissipation rate and continuous initial condition g, then

$$u(t,x) = \mathrm{E}_x\left[g(B(t))\mathbf{1}(t<\tau)\right],$$

where τ is the first exit time from the domain U.

Heat equation

Proof.

- Let $K \subset U$ be compact and let σ be the first exit time from K.
- Fixing $t > 0$ and applying Itô's formula with $f(x, y) = u(t - y, x)$ and $\zeta(s) = s$ gives, for $s < t$

$$u(t - s \wedge \sigma, B(s \wedge \sigma)) - u(t, B(0)) = \int_0^{s\wedge\sigma} \nabla_x u(t - v, B(v)) \cdot dB(v) - \int_0^{s\wedge\sigma} \partial_t u(t - v, B(v)) dv + \frac{1}{2} \int_0^{s\wedge\sigma} \Delta_x u(t - v, B(v)) dv.$$

- Since u solves the heat equation, the latter two terms cancel. Take expectations, which eliminates the remaining stochastic integral, leaving

$$\mathrm{E}_x \left[u(t - s \wedge \sigma, B(s \wedge \sigma))\right] = \mathrm{E}_x \left[u(t, B(0))\right] = u(t, x).$$

$\square$

Heat equation

Proof.

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$$- \int_0^{s \wedge \sigma} \partial_t u(t - v, B(v)) dv + \frac{1}{2} \int_0^{s \wedge \sigma} \Delta_x u(t - v, B(v)) dv.$$

- Since u solves the heat equation, the latter two terms cancel. Take expectations, which eliminates the remaining stochastic integral, leaving

$$\mathrm{E}_x \left[u(t - s \wedge \sigma, B(s \wedge \sigma)) \right] = \mathrm{E}_x \left[u(t, B(0)) \right] = u(t, x).$$

$\square$

Heat equation

Proof.

- Exhaust U by compact sets, so that $\sigma \uparrow \tau$, which gives

$$\mathrm{E}_x[u(t-s, B(s))\mathbf{1}(s<\tau)] = u(t,x).$$

Now let $t \uparrow s$.

$\square$

Heat equation

Let $\Phi(x)$ be the distribution function of a standard normal distribution.

Theorem

Let $0 < x < a$. *Then*

$$\begin{aligned}
&\mathrm{Prob}_x(B(s) \in (0,a),\ \forall\, 0 \leqslant s \leqslant t) \\
&= \sum_{k=-\infty}^{\infty} \left(\Phi\left(\frac{2ka + a - x}{\sqrt{t}}\right) - \Phi\left(\frac{2ka - x}{\sqrt{t}}\right) \right. \\
&\qquad \left. - \Phi\left(\frac{2ka + a + x}{\sqrt{t}}\right) + \Phi\left(\frac{2ka + x}{\sqrt{t}}\right) \right).
\end{aligned}$$

Proof.

- Letting $U = (0, a)$ and $g = 1$, it suffices to show that the series solves the heat equation.
- The series vanishes at 0 and a, hence satisfies the boundary condition.
- Since
$$\partial_t \Phi\left(\frac{2ka + a - x}{\sqrt{t}}\right) = \frac{1}{2}\partial_{xx}\Phi\left(\frac{2ka + a - x}{\sqrt{t}}\right)$$
the sum satisfies the heat equation.
- To check the initial condition, let $t \downarrow 0$. All but $k = 0$ terms vanish. The $k = 0$ term tends to 1.

□

Theorem

Let $d \geqslant 3$ and $V : \mathbb{R}^d \to [0, \infty)$ be bounded. Define

$$h(x) := \mathrm{E}_x \left[\exp\left(- \int_0^\infty V(B(t))dt \right) \right].$$

Then $h : \mathbb{R}^d \to [0, \infty)$ satisfies the equation

$$h(x) = 1 - \int G(x, y) V(y) h(y) dy$$

for all $x \in \mathbb{R}^d$.

Heat equation

Proof.

Define

$$R_\lambda^V f(x) := \int_0^\infty e^{-\lambda t}\,\mathrm{E}_x[f(B(t))e^{-\int_0^t V(B(s))ds}]dt.$$

Calculate

$$R_\lambda^0 f(x) - R_\lambda^V f(x) = \mathrm{E}_x \int_0^\infty e^{-\lambda t - \int_0^t V(B(s))ds} f(B(t))(e^{\int_0^t V(B(s))ds} - 1)dt$$

$$= \mathrm{E}_x \int_0^\infty e^{-\lambda t - \int_0^t V(B(s))ds} f(B(t)) \int_0^t V(B(s)) e^{\int_0^s V(B(r))dr} dsdt$$

$$= \mathrm{E}_x \int_0^\infty e^{-\lambda s} V(B(s)) \int_0^\infty e^{-\lambda t - \int_0^t V(B(s+u))du} f(B(s+t))dtds$$

$$= \mathrm{E}_x \int_0^\infty e^{-\lambda s} V(B(s)) R_\lambda^V f(B(s))ds = R_\lambda^0(V R_\lambda^V f)(x).$$

□

Heat equation

Proof.

We have

$$h(x) = \lim_{\lambda \downarrow 0} \lambda R_\lambda^V 1(x).$$

Since $R_\lambda^0 1 = \frac{1}{\lambda}$, we obtain

$$1 - \lambda R_\lambda^V 1 = \lambda R_\lambda^0 (V R_\lambda^V 1).$$

Letting $\lambda \downarrow 0$,

$$1 - h(x) = R_0^0(Vh)(x) = \int G(x,y)V(y)h(y)dy.$$

□

Occupation time

Theorem

For a standard Brownian motion $\{B(t) : t \geqslant 0\}$ *in dimension 3, let* $T = \int_0^\infty \mathbf{1}(|B(t)| < 1)dt$ *be the total occupation time of the unit ball. Then*

$$\mathrm{E}\left[e^{-\lambda T}\right] = \mathrm{sech}(\sqrt{2\lambda}).$$

Proof.

- Let $V(x) = \lambda \mathbf{1}_{B(0,1)}$ and define $h(x) = \mathrm{E}_x\left[e^{-\lambda T}\right]$.
- By the previous theorem

$$h(x) = 1 - \lambda \int_{B(0,1)} G(x,y)h(y)dy.$$

Using the classical formula for the Green's function,

$$1 - h(x) = \frac{\lambda}{2\pi|x|}\int_{B(0,|x|)} h(y)dy + \lambda \int_{B(0,1)\setminus B(0,|x|)} \frac{h(y)}{2\pi|y|}dy.$$

$\square$

Proof.

- Set $u(r) = rh(x)$ for $|x| = r$ to obtain

$$r - u(r) = 2\lambda \int_0^r su(s)ds + 2\lambda r \int_r^1 u(s)ds$$

so u solves the ODE $u'' = 2\lambda u$.

- Inserting the initial condition one can solve to find $h(0) = \operatorname{sech}(\sqrt{2\lambda})$.

$\square$