Math 639: Lecture 19

Harmonic functions and applications

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This lecture follows Mörters and Peres, Chapter 3.

Harmonic functions

Definition

By a *domain* we mean a connected open set $U \subset \mathbb{R}^d$. Let $U \subset \mathbb{R}^d$ be a domain. A function $u : U \to \mathbb{R}$ is *harmonic* if it is twice continuously differentiable and, for any $x \in U$,

$$\Delta u(x) := \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2}(x) = 0.$$

If, instead, $\Delta u \geqslant 0$ then u is *subharmonic*.

Harmonic functions

The following theorem relates harmonicity to mean value properties.

Theorem

Let $U \subset \mathbb{R}^d$ be a domain and $u : U \to \mathbb{R}$ measurable and locally bounded. The following conditions are equivalent:

1. *u is a harmonic*
2. *For any ball $B(x, r) \subset U$, we have*

$$u(x) = \frac{1}{\text{meas}(B(x,r))} \int_{B(x,r)} u(y)dy$$

3. *For any ball $B(x, r) \subset U$,*

$$u(x) = \frac{1}{\sigma_{x,r}(\partial B(x,r))} \int_{\partial B(x,r)} u(y)d\sigma_{x,r}(y)$$

where $\sigma_{x,r}$ is the surface measure on $\partial B(x, r)$.

Harmonic functions

Proof.

- Either 2 or 3 implies that one may write, for a suitable C^∞ function g of compact support,

$$u(x) = \int u(y) g(\|x - y\|_2^2) dy.$$

  Differentiating g proves that u is C^∞.
- To prove $2 \Rightarrow 3$, differentiate in the radial direction. To prove $3 \Rightarrow 2$, integrate.

□

Proof.

- To prove $3 \Leftrightarrow 1$, introduce

$$\psi(r) = \int_{\partial B(0,1)} u(x+ry)\,d\sigma_{0,1}(y)$$

and differentiate in r, applying Green's theorem, to find

$$\psi'(r) = \int_{\partial B(0,1)} \frac{\partial u}{\partial n}(x+ry)\,d\sigma_{0,1}(y) = \int_{B(0,1)} \Delta u(x+ry)\,dy,$$

which suffices.

$\square$

Maximum principle

Theorem (Maximum principle)

Suppose $u : \mathbb{R}^d \to \mathbb{R}$ is a function, which is subharmonic on an open set $U \subset \mathbb{R}^d$.

1. *If u attains its maximum in U, then u is a constant.*
2. *If u is continuous on $\overline{U}$ and U is bounded, then*

$$\max_{x \in \overline{U}} u(x) = \max_{x \in \partial U} u(x).$$

Maximum principle

Proof.

1. A variant of the argument giving the mean value characterization shows that
$$u(x) \leqslant \frac{1}{\mathrm{meas}(B(x,r))} \int_{B(x,r)} u(y)dy.$$
Hence if x is a maximum, then u is equal to this maximum on all balls containing x. Since U is connected, u is constant.
2. Since u is continuous and $\overline{U}$ is closed and bounded, the maximum of u is attained. By the previous part, the maximum is attained on ∂U.

□

Maximum principle

Corollary

Suppose $u_1, u_2 : \mathbb{R}^d \to \mathbb{R}$ *are functions which are harmonic on a bounded domain* $U \subset \mathbb{R}^d$ *and continuous on* $\overline{U}$*. If* u_1 *and* u_2 *agree on* ∂U *then they are identical on* U*.*

Brownian motion

Theorem

Suppose U is a domain, $\{B(t) : t \geqslant 0\}$ a Brownian motion started inside U and $\tau = \tau(\partial U) = \min\{t \geqslant 0 : B(t) \in \partial U\}$ the first hitting time of its boundary. Let $\phi : \partial U \to \mathbb{R}$ be measurable, and such that the function $u : U \to \mathbb{R}$ with

$$u(x) = \mathrm{E}_x[\phi(B(\tau))\mathbf{1}(\tau < \infty)], \qquad x \in U$$

is locally bounded. Then u is a harmonic function.

Proof.

For a ball $B(x,\delta) \subset U$ let $\tilde{\tau} = \inf\{t > 0 : B(t) \notin B(x,\delta)\}$. The strong Markov property implies

$$
\begin{aligned}
u(x) &= \mathrm{E}_x[\mathrm{E}_x[\phi(B(\tau))\mathbf{1}(\tau < \infty)|\mathscr{F}^+(\tilde{\tau})]] = \mathrm{E}_x[u(B(\tilde{\tau}))] \\
&= \int_{\partial B(x,\delta)} u(y)\omega_{x,\delta}(dy)
\end{aligned}
$$

where $\omega_{x,\delta}$ is the uniform measure on $\partial B(x,\delta)$. Thus u has the mean value property, and as it is locally bounded, it is harmonic. □

Dirichlet problem

Definition

Let U be a domain in $\mathbb{R}^d$ and let ∂U be its boundary. Suppose $\phi : \partial U \to \mathbb{R}$ is continuous. A continuous function $v : \overline{U} \to \mathbb{R}$ is a *solution to the Dirichlet problem* with boundary value ϕ, if it is harmonic on U and $v(x) = \phi(x)$ for $x \in \partial U$.

Poincaré cone

Definition

Let $U \subset \mathbb{R}^d$ be a domain. We say that U satisfies the *Poincaré cone condition* at $x \in \partial U$ if there exists a cone V based at x with opening angle $\alpha > 0$, and $h > 0$ such that $V \cap B(x, h) \subset U^c$.

Poincaré cone

For any open or closed set $A \subset \mathbb{R}^d$, denote by $\tau(A)$ the first hitting time of Brownian motion to A,

$$\tau(A) = \inf\{t \geqslant 0 : B(t) \in A\}.$$

Indicate by $C_z(\alpha)$ a cone of angle α with base z.

Lemma

Let $0 < \alpha < 2\pi$ *and let*

$$a = \sup_{x \in \overline{B(0,1/2)}} \mathrm{Prob}_x(\tau(\partial B(0,1)) < \tau(C_0(\alpha))).$$

Then $a < 1$ *and, for any positive integer* k *and* $h' > 0$*, we have*

$$\mathrm{Prob}_x(\tau(\partial B(z, h')) < \tau(C_z(\alpha))) \leqslant a^k$$

for all $x, z \in \mathbb{R}^d$ *with* $|x - z| < 2^{-k} h'$.

Poincaré cone

Proof.

- Checking $a < 1$ is straightforward.
- By the strong Markov property

$$\begin{aligned}&\mathrm{Prob}_x(\tau(\partial B(0,1)) < \tau(C_0(\alpha)))\\ &\leqslant \prod_{i=0}^{k-1} \sup_{x \in B(0,2^{-k+i})} \mathrm{Prob}_x(\tau(\partial B(0,2^{-k+i+1})) < \tau(C_0(\alpha))) = a^k,\end{aligned}$$

from which the second claim follows. $\square$

Dirichlet problem

Theorem

Suppose $U \subset \mathbb{R}^d$ is a bounded domain such that every boundary point satisfies the Poincaré cone condition, and suppose ϕ is a continuous function on ∂U. Let $\tau(\partial U) = \inf\{t > 0 : B(t) \in \partial U\}$, which is an almost surely finite stopping time. Then the function $u : \overline{U} \to \mathbb{R}$ given by

$$u(x) = \mathrm{E}_x[\phi(B(\tau(\partial U)))], \; x \in \overline{U},$$

is the unique continuous function harmonic on U with $u(x) = \phi(x)$ for all $x \in \partial U$.

Dirichlet problem

Proof.

- Uniqueness, and harmonicity on the interior have already been checked, so it suffices to check that the Poincaré cone condition guarantees that u extends continuously to the boundary.
- Let $z \in \partial U$ with cone $C_z(\alpha)$ based at z with angle $\alpha > 0$ such that for some $h > 0$, $C_z(\alpha) \cap B(z, h) \subset U^c$.
- By the previous lemma, for some $a < 1$, for all positive integers k and $h' > 0$ we have

$$\mathrm{Prob}_x(\tau(\partial B(z, h')) < \tau(C_z(\alpha))) \leqslant a^k$$

for all x with $|x - z| < 2^{-k} h'$.

$\square$

Dirichlet problem

Proof.

- Given $\epsilon > 0$ there is $0 < \delta < h$ such that $|\phi(y) - \phi(z)| < \epsilon$ for all $y \in \partial U$ with $|y - z| < \delta$.
- For all $x \in \overline{U}$ with $|z - x| < 2^{-k}\delta$,

$$|u(x) - u(z)| = |\operatorname{E}_x \phi(B(\tau(\partial U))) - \phi(z)| \leqslant \operatorname{E}_x |\phi(B(\tau(\partial U))) - \phi(z)|.$$

- This is bounded by

$$2\|\phi\|_\infty \operatorname{Prob}_x(\tau(\partial B(z,\delta)) < \tau(C_z(\alpha))) + \epsilon \leqslant 2\|\phi\|_\infty a^k + \epsilon$$

from which the continuity follows.

□

Liouville's theorem

Theorem

Any bounded harmonic function in $\mathbb{R}^d$ *is constant.*

Liouville's theorem

Proof.

Let u be harmonic, bounded by M and let $x \neq y$. The claim follows on averaging over balls of radius R centered at x and y, since as $R \to \infty$, the proportion that does not overlap tends to 0. $\square$

Functions harmonic on an annulus

Let

$$A = \{x \in \mathbb{R}^d : r < |x| < R\}, \qquad 0 < r < R < \infty$$

be an annulus. A solution u to $\Delta u = 0$ on A such that $u(x) = \psi(|x|^2)$ is spherically symmetric satisfies

$$0 = \sum_{i=1}^{d} \left(\psi''\left(|x|^2\right) 4x_i^2 + 2\psi'\left(|x|^2\right)\right) = 4|x|^2\psi''(|x|^2) + 2d\psi'(|x|^2).$$

Let $y = |x|^2 > 0$ so that this becomes $\psi''(y) = \frac{-d}{2y}\psi'(y)$. This gives

$$u(x) = \begin{cases} |x| & d = 1 \\ 2\log|x| & d = 2 \\ |x|^{2-d} & d \geqslant 3 \end{cases}.$$

Write $u(|x|)$ in place of $u(x)$.

Exit times from an annulus

Theorem

Suppose $\{B(t) : t \geqslant 0\}$ *is a Brownian motion in dimension* $d \geqslant 1$ *started in* $x \in A$*, which is an open annulus* A *with radii* $0 < r < R < \infty$*. Define stopping times*

$$T_r = \tau(\partial B(0,r)) = \inf\{t > 0 : |B(t)| = r\}, \qquad r > 0$$

We have

$$\mathrm{Prob}_x(T_r < T_R) = \begin{cases} \frac{R-|x|}{R-r} & d = 1 \\ \frac{\log R - \log|x|}{\log R - \log r} & d = 2 \\ \frac{R^{2-d}-|x|^{2-d}}{R^{2-d}-r^{2-d}} & d \geqslant 3 \end{cases}.$$

For any $x \notin B(0,r)$*, we have*

$$\mathrm{Prob}_x(T_r < \infty) = \begin{cases} 1 & d \leqslant 2 \\ \frac{r^{d-2}}{|x|^{d-2}} & d \geqslant 3 \end{cases}.$$

Exit times from an annulus

Proof.

Let $T = T_r \wedge T_R$. We have

$$u(x) = \mathrm{E}_x[u(B(T))] = u(r)\,\mathrm{Prob}_x(T_r < T_R) + u(R)(1 - \mathrm{Prob}_x(T_r < T_R))$$

so

$$\mathrm{Prob}_x(T_r < T_R) = \frac{u(R) - u(x)}{u(R) - u(r)}.$$

Letting $R \to \infty$ gives the second part. $\square$

Recurrence and transience

Definition

A Markov process $\{X(t) : t \geqslant 0\}$ with values in $\mathbb{R}^d$ is

- *point recurrent* if, almost surely, for every $x \in \mathbb{R}^d$ there is a (random) sequence $t_n \uparrow \infty$ such that $X(t_n) = x$ for all $n \in \mathbb{N}$
- *neighborhood recurrent* if, almost surely, for every $x \in \mathbb{R}^d$ and $\epsilon > 0$, there exists a (random) sequence $t_n \uparrow \infty$ such that $X(t_n) \in B(x, \epsilon)$ for all $n \in \mathbb{N}$.
- *transcient* if it converges to infinity almost surely.

Recurrence and transience

Theorem

Brownian motion is

- *Point recurrent in dimension 1*
- *Neighborhood recurrent, but not point recurrent in dimension 2*
- *Transient in dimension* $d \geqslant 3$.

Recurrence and transience

Proof.

- The case $d = 1$ may be deduced from $d = 2$.
- When $d = 2$, fix $\epsilon > 0$ and $x \in \mathbb{R}^d$. By the previous theorem, the stopping time $t_1 = \inf\{t > 0 : B(t) \in B(x, \epsilon)\}$ is almost surely finite. Iterating proves the neighborhood recurrence at the point x.
- This proves the neighborhood recurrence in general since the topology has a countable base.
- Point recurrence does not hold, because a.s. Brownian motion has no area.

$\square$

Recurrence and transience

Proof.

- When $d \geqslant 3$ define event
$$A_n := \{|B(t)| > n, \text{ all } t \geqslant T_{n^3}\}.$$
Note that $T_{n^3} < \infty$ with probability 1.
- For $n \geqslant |x|^{\frac{1}{3}}$,
$$\mathrm{Prob}_x(A_n^c) = \mathrm{E}_x\left[\mathrm{Prob}_{B(T_{n^3})}\{T_n < \infty\}\right] = \left(\frac{1}{n^2}\right)^{d-2}.$$
- The RHS is summable, so by Borel-Cantelli, occurs only finitely often with probability 1.

□

Dvoretzky-Erdős test

Theorem

Let $\{B(t) : t \geqslant 0\}$ be Brownian motion in $\mathbb{R}^d$ for $d \geqslant 3$ and $f : (0, \infty) \to (0, \infty)$ increasing. Then

$$\int_1^\infty f(r)^{d-2} r^{-\frac{d}{2}} dr < \infty \iff \liminf_{t \uparrow \infty} \frac{|B(t)|}{f(t)} = \infty \text{ a.s.}$$

Conversely, if the integral diverges, then $\liminf_{t\uparrow\infty} \frac{|B(t)|}{f(t)} = 0$ *a.s.*

Dvoretzky-Erdős test

We use several lemmas from homework.

Lemma (Paley-Zygmund inequality)

For any non-negative random variable X with $0 < \mathrm{E}[X^2] < \infty$,

$$\mathrm{Prob}(X > 0) \geqslant \frac{\mathrm{E}[X]^2}{\mathrm{E}[X^2]}.$$

Lemma (Borel-Cantelli)

Suppose $E_1, E_2, \ldots$ are events with

$$\sum_{n=1}^{\infty} \mathrm{Prob}(E_n) = \infty, \qquad \liminf_{k\to\infty} \frac{\sum_{m=1}^{k}\sum_{n=1}^{k} \mathrm{Prob}(E_n \cap E_m)}{\left(\sum_{n=1}^{k} \mathrm{Prob}(E_n)\right)^2} < \infty.$$

Then with positive probability infinitely many of the events take place.

Dvoretzky-Erdős test

Lemma

There exists a constant $C_1 > 0$ depending only on the dimension d such that, for any $\rho > 0$, we have

$$\sup_{x \in \mathbb{R}^d} \mathrm{Prob}_x(\textit{there exists } t > 1 \textit{ with } |B(t)| \leqslant \rho) \leqslant C_1 \rho^{d-2}.$$

Proof.

Calculate

$$\mathrm{Prob}_x(\text{there exists } t > 1 \text{ with } |B(t)| \leqslant \rho) \leqslant \mathrm{E}_0\left[\left(\frac{\rho}{|B(1)+x|}\right)^{d-2}\right]$$

$$\leqslant \rho^{d-2}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}|y+x|^{2-d}\exp\left(-\frac{|y|^2}{2}\right)dy.$$

$\square$

Dvoretzky-Erdős test

Proof of the Dvoretzky-Erdős test.

- Define events

$$A_n = \{\text{there exists } t \in (2^n, 2^{n+1}] \text{ with } |B(t)| \leqslant f(t)\}.$$

- We have

$$\begin{aligned}\mathrm{Prob}(A_n) &\leqslant \mathrm{Prob}(\text{there exists } t > 1 \text{ with } |B(t)| \leqslant f(2^{n+1})2^{-n/2}) \\ &\leqslant C_1(f(2^{n+1})2^{-n/2})^{d-2}.\end{aligned}$$

$\square$

Dvoretzky-Erdős test

Proof of the Dvoretzky-Erdős test.

- Convergence of the integral is equivalent to convergence of

$$\sum_{n=1}^{\infty} \left(f(2^n)2^{-n/2}\right)^{d-2} < \infty.$$

- By Borel-Cantelli, the events A_n happen only finitely often with probability 1. This holds replacing f with a constant multiple, so

$$\liminf_{t\uparrow\infty} \frac{|B(t)|}{f(t)} = \infty$$

with probability 1.

□

Proof of the Dvoretzky-Erdős test.

- Now suppose
$$\sum_{n=1}^{\infty}\left(f(2^n)2^{-n/2}\right)^{d-2} = \infty.$$
and assume, as we may, that $f(t) < \sqrt{t}$.
- For $\rho \in (0,1)$, consider the random variable $I_\rho = \int_1^2 \mathbf{1}(|B(t)| \leqslant \rho)dt$. One has
$$C_2\rho^d \leqslant \mathrm{E}[I_\rho] \leqslant C_3\rho^d.$$

$\square$

Dvoretzky-Erdős test

Proof of the Dvoretzky-Erdős test.

- Estimate

$$\begin{aligned} \mathrm{E}[I_\rho^2] &= 2\,\mathrm{E}\left[\int_1^2 \mathbf{1}(|B(t)| \leqslant \rho) \int_t^2 \mathbf{1}(|B(s)| \leqslant \rho) ds dt\right] \\ &\leqslant 2\,\mathrm{E}\left[\int_1^2 \mathbf{1}(|B(t)| \leqslant \rho)\, \mathrm{E}_{B(t)} \int_0^\infty \mathbf{1}(|\tilde{B}(s)| \leqslant \rho) ds dt\right]. \end{aligned}$$

- Given $x \neq 0$, let $T = \inf\{t > 0 : |B(t)| = x\}$ and use the strong Markov property to obtain

$$\begin{aligned} \mathrm{E}_0 \int_0^\infty \mathbf{1}(|B(s)| \leqslant \rho) ds \geqslant \mathrm{E} \int_T^\infty \mathbf{1}(|B(s)| \leqslant \rho) ds \\ = \mathrm{E}_x \int_0^\infty \mathbf{1}(|B(s)| \leqslant \rho) ds \end{aligned}$$

□

Proof of the Dvoretzky-Erdős test.

- Thus
$$\mathrm{E}[I_\rho^2] \leqslant 2C\rho^d\, \mathrm{E}_0 \int_0^\infty \mathbf{1}(|B(s)| \leqslant \rho) ds \leqslant C'\rho^{d+2}.$$
- It follows that
$$\mathrm{Prob}(I_\rho > 0) \geqslant \frac{\mathrm{E}[I_\rho]^2}{\mathrm{E}[I_\rho^2]} \geqslant C''\rho^{d-2}.$$
- Choose $\rho = f(2^n)2^{-n/2}$. By Brownian scaling and monotonicity of f,
$$\mathrm{Prob}(A_n) \geqslant \mathrm{Prob}(I_\rho > 0) \geqslant C'' \left(f(2^n)2^{-n/2}\right)^{d-2}$$
so $\sum_n \mathrm{Prob}(A_n) = \infty$.

□

Dvoretzky-Erdős test

Proof of the Dvoretzky-Erdős test.

- For $m < n - 1$,

$$\begin{aligned}\mathrm{Prob}[A_n|A_m] &\leqslant \sup_{x\in\mathbb{R}^d} \mathrm{Prob}_x(\exists\ t > 1 \text{ with } |B(t)| \leqslant f(2^{n+1})2^{(1-n)/2})\\ &\leqslant C_1(f(2^{n+1})2^{(1-n)/2})^{d-2}.\end{aligned}$$

$\square$

Proof of the Dvoretzky-Erdős test.

- Thus

$$\liminf_{k\to\infty} \frac{\sum_{m=1}^k \sum_{n=1}^k \mathrm{Prob}(A_n \cap A_m)}{\left(\sum_{n=1}^k \mathrm{Prob}(A_n)\right)^2}$$

$$= 2 \liminf_{k\to\infty} \frac{\sum_{m=1}^k \mathrm{Prob}(A_m) \sum_{n=m+2}^k \mathrm{Prob}(A_n | A_m)}{\left(\sum_{n=1}^k \mathrm{Prob}(A_n)\right)^2}$$

$$\ll \liminf_{k\to\infty} \frac{\sum_{n=1}^k (f(2^{n+1})2^{(1-n)/2})^{d-2}}{\sum_{n=1}^k (f(2^n)2^{-n/2})^{d-2}} < \infty.$$

- We thus get $\mathrm{Prob}(A_n\ i.o.) > 0$, so it is 1 since it has a 0-1 law. Since we can replace f with ϵf, $\liminf_{t\uparrow\infty} \frac{|B(t)|}{f(t)} = 0$ a.s.

$\square$

Occupation measures

Theorem

Let $\{B(s) : s \geqslant 0\}$ be a linear Brownian motion and $t > 0$. Define the occupation measure μ_t by

$$\mu_t(A) = \int_0^t \mathbf{1}_A(B(s))ds, \qquad A \subset \mathbb{R} \text{ Borel.}$$

Then a.s. μ_t is absolutely continuous with respect to Lebesgue measure.

Occupation measures

Proof.

It suffices to check that

$$\liminf_{r\downarrow 0} \frac{\mu_t(B(x,r))}{\mathrm{meas}(B(x,r))} < \infty, \;\; \mu_t - a.e.\; x \in \mathbb{R}.$$

By Fatou and Fubini,

$$\begin{aligned} \mathrm{E} \int \liminf_{t\downarrow 0} \frac{\mu_t(B(x,r))}{\mathrm{meas}(B(x,r))} &\leqslant \liminf_{r\downarrow 0} \frac{1}{2r} \mathrm{E} \int \mu_t(B(x,r)) d\mu_t(x) \\ &= \liminf_{r\downarrow 0} \frac{1}{2r} \int_0^t \int_0^t \mathrm{Prob}(|B(s_1) - B(s_2)| \leqslant r) ds_1 ds_2. \end{aligned}$$

□

Occupation measures

Proof.

Use

$$\mathrm{Prob}(|B(s_1) - B(s_2)| \leqslant r) = \mathrm{Prob}\left(|X| \leqslant \frac{r}{\sqrt{|s_1 - s_2|}}\right) \leqslant \frac{2r}{\sqrt{|s_1 - s_2|}}$$

which implies that

$$\liminf_{r \downarrow 0} \frac{1}{2r} \int_0^t \int_0^t \mathrm{Prob}(|B(s_1) - B(s_2)| \leqslant r) ds_1 ds_2 \leqslant \int_0^t \int_0^t \frac{ds_1 ds_2}{\sqrt{|s_1 - s_2|}} < \infty.$$

□

Occupation measures

Theorem

Let $U \subset \mathbb{R}^d$ *be a non-empty bounded open set and* $x \in \mathbb{R}^d$ *arbitrary.*

- *If* $d = 2$, *then* Prob_x*-a.s.*, $\int_0^\infty \mathbf{1}_U(B(t))dt = \infty$.
- *If* $d \geqslant 3$, *then* $\mathrm{E}_x \int_0^\infty \mathbf{1}_U(B(t))dt < \infty$.

Occupation measures

Proof.

- It suffices to show this for balls, and by translation, for balls centered at 0. Let $U = B(0, r)$.
- First consider $d = 2$ and let $G = B(0, 2r)$.
- Let $S_0 = 0$ and, for all $k \geqslant 0$, let

$$T_k = \inf\{t > S_k : B(t) \notin G\}, \qquad S_{k+1} = \inf\{t > T_k : B(t) \in U\}.$$

□

Occupation measures

Proof.

- By the strong Markov property

$$
\begin{aligned}
&\mathrm{Prob}_x\left(\int_{S_k}^{T_k} \mathbf{1}_U(B(t))dt \geqslant s \,\middle|\, \mathscr{F}^+(S_k)\right)\\
&= \mathrm{Prob}_{B(S_k)}\left(\int_0^{T_1} \mathbf{1}_U(B(t))dt \geqslant s\right)\\
&= \mathrm{E}_x\left[\mathrm{Prob}_{B(S_k)}\left(\int_0^{T_1} \mathbf{1}_U(B(t))dt \geqslant s\right)\right]\\
&= \mathrm{Prob}_x\left(\int_{S_k}^{T_k} \mathbf{1}_U(B(t))dt \geqslant s\right).
\end{aligned}
$$

These variables are i.i.d. with positive mean, so the conclusion follows by the strong law of large numbers.

□

Occupation measures

Proof.

- Now let $d \geqslant 3$. Write $p(\cdot, \cdot, \cdot)$ for the transition kernel of Brownian motion. By Fubini's theorem,

$$\begin{aligned}
&\mathrm{E}_0 \int_0^\infty \mathbf{1}_{B(0,r)}(B(s))ds = \int_0^\infty \mathrm{Prob}_0(B(s) \in B(0,r))ds \\
&= \int_0^\infty \int_{B(0,r)} p(s,0,y)dyds \\
&= \int_{B(0,r)} \int_0^\infty p(s,0,y)dsdy \\
&= \sigma(\partial B(0,1)) \int_0^r \rho^{d-1} \int_0^\infty \left(\frac{1}{\sqrt{2\pi s}}\right)^d e^{-\frac{\rho^2}{2s}} dsd\rho \\
&= C \int_0^r \rho^{d-1}\rho^{2-d} d\rho < \infty.
\end{aligned}$$

$\square$

Proof.

- To handle a general starting point x, let T be the stopping time for Brownian motion started at 0 and stopped the first time it reaches a sphere of radius $|x|$. Then

$$\begin{aligned}\mathrm{E}_x \int_0^\infty \mathbf{1}_{B(0,r)}(B(s))ds &= \mathrm{E}_0 \int_T^\infty \mathbf{1}_{B(0,r)}(B(s))ds \\ &\leqslant \mathrm{E}_0 \int_0^\infty \mathbf{1}_{B(0,r)}(B(s))ds < \infty.\end{aligned}$$

□

Transient Brownian motion

Definition

Suppose that $\{B(t) : 0 \leqslant t \leqslant T\}$ is a d-dimensional Brownian motion and one of the following three cases holds:

1. $d \geqslant 3$ and $T = \infty$
2. $d \geqslant 2$ and T is an independent exponential time with parameter $\lambda > 0$,
3. $d \geqslant 2$ and T is the first exit time from a bounded domain D.

Say $D = \mathbb{R}^d$ in cases 1 and 2. We refer to these three cases by saying that $\{B(t) : 0 \leqslant t \leqslant T\}$ is a *transient Brownian motion*.

Transient Brownian motion

Theorem

For transient Brownian motion $\{B(t) : 0 \leqslant t \leqslant T\}$ there exists a transition density $p^* : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ such that, for any $t > 0$,

$$\mathrm{Prob}_x(B(t) \in A, t \leqslant T) = \int_A p^*(t, x, y)dy, \quad \forall A \in \mathscr{B}(\mathbb{R}^d).$$

Moreover, for all $t \geqslant 0$ and a.e. $x, y \in D$, $p^*(t, x, y) = p^*(t, y, x)$.

For the proof, see MP p. 79.

Transient Brownian motion

We make the following convention regarding transition kernels for transient Brownian motion.

1. $d \geqslant 3$ and $T = \infty$:
$$p^*(t, x, y) = p(t, x, y).$$
2. $d \geqslant 2$ and T is an independent exponential time with parameter $\lambda > 0$:
$$p^*(t, x, y) = e^{-\lambda t} p(t, x, y).$$
3. $d \geqslant 2$ and T is the first exit time from a bounded domain D:
$$p^*(t, x, y) = p(t, x, y) - \mathrm{E}_x[p(t - T, B(T), y) 1(T < t)].$$

Green's function

Definition

For transient Brownian motion $\{B(t) : 0 \leqslant t \leqslant T\}$ we define the *Green's function* $G : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ by

$$G(x, y) = \int_0^\infty p^*(t, x, y)\,dt.$$

Green's function

Theorem

If $f : \mathbb{R}^d \to [0, \infty]$ *is measurable, then*

$$\mathrm{E}_x \int_0^T f(B(t))dt = \int f(y)G(x, y)dy.$$

Proof.

Fubini gives

$$\begin{aligned}\mathrm{E}_x \int_0^T f(B(t))dt &= \int_0^\infty \mathrm{E}_x[f(B(t))1_{(t\leqslant T)}]dt = \int_0^\infty \int p^*(t, x, y)f(y)dydt \\ &= \int \int_0^\infty p^*(t, x, y)dtf(y)dy = \int G(x, y)f(y)dy.\end{aligned}$$

$\square$

Green's function

Theorem

If $d \geqslant 3$ and $T = \infty$, then

$$G(x,y) = c(d)|x-y|^{2-d}, \qquad c(d) = \frac{\Gamma(d/2-1)}{2\pi^{d/2}}.$$

Proof.

Calculate

$$\begin{aligned} G(x,y) &= \int_0^\infty \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2t}} dt \\ &= \frac{|x-y|^{2-d}}{2\pi^{d/2}} \int_0^\infty s^{d/2-2} e^{-s} ds = \frac{\Gamma(d/2-1)}{2\pi^{d/2}} |x-y|^{2-d}. \end{aligned}$$

$\square$

Green's function

Theorem

If $d = 2$ and T is an independent exponential time with parameter $\lambda > 0$, then

$$G(x, y) \sim -\frac{1}{\pi} \log |x - y|, \qquad |x - y| \downarrow 0.$$

See MP. p.81.

Theorem

In all three cases of transient Brownian motion in $d \geqslant 2$, the Green's function $G : D \times D \to [0, \infty]$ has the following properties:

1. *G is finite off and infinite on the diagonal $\Delta = \{(x, y) : x = y\}$.*
2. *G is symmetric, i.e. $G(x, y) = G(y, x)$ for all $x, y \in D$.*
3. *For $y \in D$ the Green's function $G(\cdot, y)$ is subharmonic on $D \setminus \{y\}$. In cases 1 and 3 it is harmonic.*

This is immediate in the case $d = 3$. In the remaining cases, see MP, pp. 82-84.

Green's function

Lemma

If $d = 2$*, for* $x, y, z \in \mathbb{R}^2$ *with* $|x - z| = 1$*,*

$$-\frac{1}{\pi} \log |x - y| = \int_0^\infty p(s, x, y) - p(s, x, z) ds.$$

Proof.

For $|x - z| = 1$, we obtain

$$\int_0^\infty p(t,x,y) - p(t,x,z)dt = \frac{1}{2\pi}\int_0^\infty \left(e^{-\frac{|x-y|^2}{2t}} - e^{-\frac{1}{2t}}\right)\frac{dt}{t}$$

$$= \frac{1}{2\pi}\int_0^\infty \left(\int_{|x-y|^2/(2t)}^{1/(2t)} e^{-s}ds\right)\frac{dt}{t}$$

$$= \frac{1}{2\pi}\int_0^\infty e^{-s}\int_{|x-y|^2/(2s)}^{1/(2s)} \frac{dt}{t}ds = -\frac{\log|x-y|}{\pi}.$$

$\square$

Harmonic measure

Definition

Let $\{B(t) : t \geqslant 0\}$ be a d-dimensional Brownian motion, $d \geqslant 2$, started in some point x and fix a closed set $A \subset \mathbb{R}^d$. Define a measure $\mu_A(x, \cdot)$ by

$$\mu_A(x, B) = \mathrm{Prob}(B(\tau) \in B, \tau < \infty), \qquad \tau = \inf\{t \geqslant 0 : B(t) \in A\}$$

for $B \subset A$ Borel.

$\mu_A(x, \cdot)$ is the distribution of the first hitting point of A, and the total mass of the measure is the probability that a Brownian motion started in x ever hits the set A. If $x \notin A$, $\mu_A(x, \cdot)$ is supported on ∂A.

Dirichlet problem

Theorem

If the Poincaré cone condition is satisfied at every point $x \in \partial U$ on the boundary of a bounded domain U, then the solution of the Dirichlet problem with boundary condition $\phi : \partial U \to \mathbb{R}$ can be written

$$u(x) = \int \phi(y)\mu_{\partial U}(x, dy), \qquad x \in \overline{U}.$$

This is a restatement of our earlier solution of the Dirichlet problem.

Harnack principle

Theorem (Harnack principle)

Suppose $A \subset \mathbb{R}^d$ is compact and x, y are in the unbounded component of A^c. Then $\mu_A(x, \cdot)$ is absolutely continuous with respect to $\mu_A(y, \cdot)$.

Harnack principle

Proof.

Given $B \subset \partial A$ Borel, the mapping $x \mapsto \mu_A(x, B)$ is a harmonic function on A^c. If it vanishes at $y \in A^c$ then this is a minimum, so the maximum modulus principle implies $\mu_A(x, B) = 0$ for all $x \in A^c$, as needed. □

Polar and nonpolar

Definition

A compact set A is called *nonpolar* if $\mu_A(x, A) > 0$ for some (all) $x \in A^c$. Otherwise it is called *polar*.

Poisson's formula

Theorem (Poisson's formula)

Suppose that $B \subset \partial B(0,1)$ is a Borel subset of the unit sphere for $d \geqslant 2$. Let ω denote the uniform distribution on the unit sphere. Then, for all $x \notin \partial B(0,1)$,

$$\mu_{\partial B(0,1)}(x, B) = \int_B \frac{|1 - |x|^2|}{|x-y|^d} d\omega(y).$$

Poisson's formula

Proof.

- We first consider the case $|x| < 1$.
- Let τ denote the first hitting time of $\partial B(0,1)$.
- It suffices by density to check for smooth f

$$\mathrm{E}_x[f(B(\tau))] = \int_{\partial B(0,1)} \frac{1 - |x|^2}{|x-y|^d} f(y) d\omega(y).$$

Thus it suffices to check that the RHS is a solution to the Dirichlet problem with boundary value f.

$\square$

Poisson's formula

Proof.

- One may check by differentiating that for all $y \in \partial B(0,1)$,

$$x \mapsto \frac{1-|x|^2}{|x-y|^d}$$

  is harmonic on the open ball $B(0,1)$. This proves the harmonicity.

- To prove the extension to the boundary first consider $f \equiv 1$ and check

$$I(x) = \int_{\partial B(0,1)} \frac{1-|x|^2}{|x-y|^d}\omega(dy) = 1.$$

  Indeed, I is harmonic on the interior, satisfies spherical symmetric, and has value 1 at 0.

$\square$

Poisson's formula

Proof.

- For general f, and $y \in \partial B(0,1)$,

$$\left| f(y) - \int_{\partial B(0,1)} \frac{1-|x|^2}{|x-z|^d} f(z) d\omega(z) \right|$$
$$= \left| \int_{\partial B(0,1)} \frac{1-|x|^2}{|x-z|^d} (f(y) - f(z)) d\omega(z) \right|.$$

- Note that $\frac{1-|x|^2}{|x-z|^d} d\omega(z)$ is a probability measure on the boundary which is a summability kernel for δ_y as $x \to y$.

$\square$

Poisson's formula

Proof.

- If $|x| > 1$ we use inversion in the unit sphere. One can check that

$$u : \overline{B(0,1)}^c \to \mathbb{R}$$

is harmonic if and only if its inversion

$$u^* : B(0,1)\backslash\{0\} \to \mathbb{R}, \; u^*(x) = u\left(\frac{x}{|x|^2}\right)|x|^{2-d}$$

is harmonic.

□

Poisson's formula

Proof.

- Given smooth f, define harmonic function $u : \overline{B(0,1)}^c \to \mathbb{R}$,

$$u(x) = \mathrm{E}_x[f(B(\tau))\mathbf{1}(\tau < \infty)].$$

  Thus u^* is bounded and harmonic, and hence has a unique extension to a harmonic function at 0, also.
- The harmonic extension is continuous on the closure, where it agrees with f, which gives the claimed formula.

$\square$

Harmonic measure

Theorem

Let $A \subset \mathbb{R}^d$ be a compact, nonpolar set, then there exists a probability measure μ_A on A given by

$$\mu_A(B) = \lim_{x \to \infty} \mathrm{Prob}_x(B(\tau(A)) \in B | \tau(A) < \infty)$$

for $B \subset A$ Borel. Moreover, if $B(x, r) \supset A$ and $\omega_{x,r}$ is the uniform probability measure on its boundary then

$$\mu_A(B) = \frac{\int \mu_A(a, B) d\omega_{x,r}(a)}{\int \mu_A(a, A) d\omega_{x,r}(a)}.$$

See MP pp.87 – 91.