# Math 639: Lecture 18

## Brownian motion as a Markov process

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# Brownian motion as a Markov process

This lecture follows Mörters and Peres, Chapter 2.

# Brownian motion as a Markov process

**Definition**

If $B_1, ..., B_d$ are Brownian motions started in $x_1, ..., x_d$, then the stochastic process $\{B(t) : t \geqslant 0\}$ given by

$$B(t) = (B_1(t), ..., B_d(t))^T$$

is called *d-dimensional Brownian motion* started in $(x_1, ..., x_d)^T$. The $d$-dimensional Brownian motion started in the origin is called *standard Brownian motion*. One dimensional Brownian motion is called *linear*, two-dimensional Brownian motion *planar Brownian motion*.

# Independent stochastic processes

**Definition**

The stochastic processes $\{X(t) : t \geqslant 0\}$ and $\{Y(t) : t \geqslant 0\}$ are called *independent*, if for any sets $t_1, ..., t_n \geqslant 0$ and $s_1, ..., s_m \geqslant 0$ of times the vectors $(X(t_1), ..., X(t_n))$ and $(Y(s_1), ..., Y(s_m))$ are independent.

# Markov property

**Theorem**

*Suppose that* $\{B(t) : t \geqslant 0\}$ *is a Brownian motion started in* $x \in \mathbb{R}^d$. *Let* $s > 0$, *then the process* $\{B(t+s) - B(s) : t \geqslant 0\}$ *is again a Brownian motion started at the origin, and is independent of the process* $\{B(t) : 0 \leqslant t \leqslant s\}$.

**Proof.**

One easily checks that the f.d.d. of the shifted Brownian motion agree with those of Brownian motion. Independence follows from the independence of increments. □

# Filtration

## Definition

1. A *filtration* on a probability space $(\Omega, \mathscr{F}, \text{Prob})$ is a family $(\mathscr{F}(t) : t \geqslant 0)$ of $\sigma$-algebras such that $\mathscr{F}(s) \subset \mathscr{F}(t) \subset \mathscr{F}$ for all $s < t$.
2. A probability space together with a filtration is called a *filtered probability space*.
3. A stochastic process $\{X(t) : t \geqslant 0\}$ defined on a filtered probability space with filtration $(\mathscr{F}(t) : t \geqslant 0)$ is called *adapted* if $X(t)$ is $\mathscr{F}(t)$-measurable for any $t \geqslant 0$.

# Filtration

Given a Brownian motion $\{B(t) : t \geqslant 0\}$ defined on some probability space, then a filtration $\mathscr{F}^0(t), t \geqslant 0$ is defined by letting

$$\mathscr{F}^0(t) = \sigma(B(s) : 0 \leqslant s \leqslant t).$$

A larger $\sigma$-algebra $\mathscr{F}^+(s)$ is defined by

$$\mathscr{F}^+(s) = \bigcap_{t>s} \mathscr{F}^0(t).$$

# Markov property

**Theorem**

*For every* $s \geqslant 0$ *the process* $\{B(t+s) - B(s) : t \geqslant 0\}$ *is independent of the* $\sigma$*-algebra* $\mathscr{F}^+(s)$.

**Proof.**

- Let $\{s_n : n \in \mathbb{N}\}$ be a monotone decreasing sequence tending to $s$.
- By continuity, for any $t_1, ..., t_m \geqslant 0$ we have

$$\begin{aligned}&(B(t_1 + s) - B(s), ..., B(t_m + s) - B(s)) \\ &= \lim_{j\uparrow\infty}(B(t_1 + s_j) - B(s_j), ..., B(t_m + s_j) - B(s_j))\end{aligned}$$

is independent of $\mathscr{F}^+(s)$. This proves independence of the process $\{B(t+s) - B(s) : t \geqslant 0\}$ with $\mathscr{F}^+(s)$.

$\square$

**Theorem (Blumenthal's 0-1 Law)**

*Let $x \in \mathbb{R}^d$ and $A \in \mathscr{F}^+(0)$. Then $\mathrm{Prob}_x(A) \in \{0, 1\}$.*

**Proof.**

Any $A \in \sigma(B(t) : t \geqslant 0)$ is independent of $\mathscr{F}^+(0)$, since the $\sigma$-algebra is generated by finite dimensional rectangles. This applies to $A \in \mathscr{F}^+(0)$, which is thus independent of itself, so that the probability is 0 or 1. $\square$

# Return to 0

**Theorem**

*Suppose* $\{B(t) : t \geqslant 0\}$ *is a 1-d Brownian motion. Define* $\tau = \inf\{t > 0 : B(t) > 0\}$ *and* $\sigma = \inf\{t > 0 : B(t) = 0\}$*. Then*

$$\mathrm{Prob}_0(\tau = 0) = \mathrm{Prob}_0(\sigma = 0) = 1.$$

# Return to 0

**Proof.**

- The event

$$\{\tau = 0\} = \bigcap_{n=1}^{\infty} \left\{ \text{there is } 0 < \epsilon < \frac{1}{n} \text{ s.t. } B(\epsilon) > 0 \right\}$$

  is in $\mathscr{F}^+(0)$, hence has probability 0 or 1.
- $\mathrm{Prob}_0(\tau \leqslant t) \geqslant \mathrm{Prob}_0(B(t) > 0) = \frac{1}{2}$ for $t > 0$, so $\mathrm{Prob}_0(\tau = 0) \geqslant \frac{1}{2}$, so the probability is 1.
- The remaining claim follows from the intermediate value theorem.

$\square$

# Maxima and minima

**Theorem**

*For a 1d Brownian motion $\{B(t) : 0 \leqslant t \leqslant 1\}$, almost surely,*

1. *Every local maximum is a strict local maximum*
2. *The set of times where the local maxima are attained is countable and dense*
3. *The global maximum is attained at a unique time*

# Maxima and minima

## Proof.

- Fix two intervals $[a_1, b_1]$, $[a_2, b_2]$, $b_1 \leqslant a_2$. Let the maxima of Brownian motion on these intervals be $m_1$ and $m_2$
- By the previous theorem, $B(a_2) < m_2$ a.s., and so the maxima on $[a_2, b_2]$ agrees with that on the interval $[a_2 - \epsilon, b_2]$ for some $\epsilon > 0$, so we may assume that $b_1 < a_2$.
- By the Markov property, $m_1 - B(b_1)$, $B(a_2) - B(b_1)$, and $m_2 - B(a_2)$ are independent.
- Write the event $m_1 = m_2$ as

$$B(a_2) - B(b_1) = m_1 - B(b_1) - (m_2 - B(a_2)).$$

Conditioned on $m_1 - B(b_1)$ and $m_2 - B(a_2)$, the right hand side is constant, while the left hand side is normally distributed, so that the equality has measure 0. $\square$

# Maxima and minima

**Proof.**

1. To verify that all local maxima are strict, note almost surely that the maxima differ over any two non-overlapping rational intervals
2. Almost surely, there is a strict local maximum in the interior of each closed bounded interval with distinct rational endpoints. Hence these are dense, and their number is countable.
3. Almost surely, for any rational $q$, the maxima in $[0, q]$ and $[q, 1]$ are different. If there are two points of a global maxima $t_1 < t_2$ then there is a rational $q$, $t_1 < q < t_2$, so this happens with measure 0.

$\square$

# Stopping times

**Definition**

A random variable $T$ with values in $[0, \infty]$, defined on a probability space with filtration $(\mathscr{F}(t) : t \geqslant 0)$ is called a *stopping time* with respect to $(\mathscr{F}(t) : t \geqslant 0)$ if $\{T \leqslant t\} \in \mathscr{F}(t)$, for every $t \geqslant 0$.

# Stopping times

- If $(T_n : n = 1, 2, \ldots)$ is an increasing sequence of stopping times with respect to $(\mathscr{F}_(t) : t \geqslant 0)$ and $T_n \uparrow T$, then $T$ is a stopping time w.r.t. $(\mathscr{F}(t) : t \geqslant 0)$, since

$$\{T \leqslant t\} = \bigcap_{n=1}^{\infty} \{T_n \leqslant t\} \in \mathscr{F}(t).$$

- Let $T$ be a stopping time w.r.t. $(\mathscr{F}(t) : t \geqslant 0)$. Define

$$T_n = \frac{m+1}{2^n}, \; \frac{m}{2^n} \leqslant T < \frac{m+1}{2^n}.$$

  This is a stopping time.

# Stopping times

- Every stopping time w.r.t. $(\mathscr{F}^0(t) : t \geqslant 0)$ is also a stopping time w.r.t. $(\mathscr{F}^+(t) : t \geqslant 0)$, since $\mathscr{F}^0(t) \subset \mathscr{F}^+(t)$.
- Let $H$ be a closed set. The first hitting time to $H$, $T = \inf\{t \geqslant 0 : B(t) \in H\}$ of the set $H$ is a stopping time w.r.t. $(\mathscr{F}^0(t) : t \geqslant 0)$. Note

$$\{T \leqslant t\} = \bigcap_{n=1}^{\infty} \bigcup_{s \in \mathbb{Q} \cap (0,t)} \bigcup_{x \in \mathbb{Q}^d \cap H} \left\{ |B(s) - x| \leqslant \frac{1}{n} \right\} \in \mathscr{F}^0(t).$$

# Stopping times

- Let $G \subset \mathbb{R}^d$ be open, then

$$T = \inf\{t \geqslant 0 : B(t) \in G\}$$

is a stopping time w.r.t. filtration $(\mathscr{F}^+(t) : t \geqslant 0)$, but not necessarily w.r.t $(\mathscr{F}^0(t) : t \geqslant 0)$. To see the first claim, write

$$\{T \leqslant t\} = \bigcap_{s>t} \{T < s\} = \bigcap_{s>t} \bigcup_{r \in \mathbb{Q} \cap (0,s)} \{B(r) \in G\} \in \mathscr{F}^+(t).$$

# Stopping times

- To see the second claim, let $G$ be a open half-space and suppose the starting point is not in $\overline{G}$.
- Let $\gamma : [0, t] \to \mathbb{R}^d$ with $\gamma(0, t) \cap \overline{G} = \varnothing$ and $\gamma(t) \in \partial G$.
- The $\sigma$-algebra $\mathscr{F}^0(t)$ contains no non-trivial subset of $\{B(s) = \gamma(s), 0 \leqslant s \leqslant t\}$. If $\{T \leqslant t\} \in \mathscr{F}^0(t)$, the set

$$\{B(s) = \gamma(s), 0 \leqslant s \leqslant t, T = t\}$$

would be in $\mathscr{F}^0(t)$ and a non-trivial subset of the earlier set.

## Stopping times

- We make the convention that stopping times are defined w.r.t. $(\mathscr{F}^+(t), t \geqslant 0)$
- The filtration $(\mathscr{F}^+(t), t \geqslant 0)$ satisfies *right-continuity*,

$$\bigcap_{\epsilon > 0} \mathscr{F}^+(t + \epsilon) = \mathscr{F}^+(t).$$

# Stopping times

## Lemma

*Suppose a random variable $T$ with values in $[0, \infty]$ satisfies $\{T < t\} \in \mathscr{F}(t)$, for every $t \geqslant 0$, and $(\mathscr{F}(t) : t \geqslant 0)$ is right-continuous, then $T$ is a stopping time w.r.t. $(\mathscr{F}(t) : t \geqslant 0)$.*

## Proof.

We have

$$\{T \leqslant t\} = \bigcap_{k=1}^{\infty} \left\{T < t + \frac{1}{k}\right\} \in \bigcap_{n=1}^{\infty} \mathscr{F}\left(t + \frac{1}{n}\right) = \mathscr{F}(t).$$

□

# Stopping times

**Definition**

Let $T$ be a stopping time. The $\sigma$-algebra generated by $T$ is

$$\mathscr{F}^+(T) = \{A \in \mathscr{A} : \forall t \geqslant 0,\ A \cap \{T \leqslant t\} \in \mathscr{F}^+(t)\}.$$

# Strong Markov property

**Theorem (Strong Markov property)**

*For every almost surely finite stopping time $T$, the process $\{B(T+t) - B(T) : t \geqslant 0\}$ is a standard Brownian motion independent of $\mathscr{F}^+(T)$.*

Alternatively, for any bounded measurable $f : C([0,\infty), \mathbb{R}^d) \to \mathbb{R}$, and $x \in \mathbb{R}^d$,

$$\mathrm{E}_x[f(\{B(T+t) : t \geqslant 0\}) | \mathscr{F}^+(T)] = \mathrm{E}_{B(T)}[f(\{\tilde{B}(t) : t \geqslant 0\})].$$

where $\tilde{B}(t)$ denotes Brownian motion started from $B(T)$.

# Strong Markov property

**Proof.**

- Set
$$T_n = (m+1)2^{-n}, \text{ if } m2^{-n} \leqslant T < (m+1)2^{-n}.$$
- Write $B_k = \{B_k(t) : t \geqslant 0\}$ for $B_k(t) = B(t + k2^{-n}) - B(k2^{-n})$. Set $B_*(t) = B(t + T_n) - B(T_n)$.
- Let $E \in \mathscr{F}^+(T_n)$. For every event $\{B_* \in A\}$, we have

$$\begin{aligned}\mathrm{Prob}(\{B_* \in A\} \cap E) &= \sum_{k=0}^{\infty} \mathrm{Prob}(\{B_k \in A\} \cap E \cap \{T_n = k2^{-n}\}) \\ &= \sum_{k=0}^{\infty} \mathrm{Prob}(B_k \in A)\,\mathrm{Prob}(E \cap \{T_n = k2^{-n}\})\end{aligned}$$

since $E \cap \{T_n = k2^{-n}\} \in \mathscr{F}^+(k2^{-n})$.

$\square$

# Strong Markov property

**Proof.**

- We have $\text{Prob}(B_k \in A) = \text{Prob}(B \in A)$ so that

$$\begin{aligned}\text{Prob}(\{B_* \in A\} \cap E) &= \text{Prob}(B \in A) \sum_{k=0}^{\infty} \text{Prob}(E \cap \{T_n = k2^{-n}\}) \\ &= \text{Prob}(B \in A)\,\text{Prob}(E).\end{aligned}$$

Thus $B_*$ is a Brownian motion which is independent of $E$, hence of $\mathscr{F}^+(T_n)$.

□

# Strong Markov property

**Proof.**

- As $T_n \downarrow T$, we have $\{B(s+T_n) - B(T_n) : s \geqslant 0\}$ is a Brownian motion independent of $\mathscr{F}^+(T_n) \supset \mathscr{F}^+(T)$. Hence the increments

  $$B(s+t+T) - B(t+T) = \lim_{n\to\infty} B(s+t+T_n) - B(t+T_n)$$

  so that the increments of the process $\{B(r+T) - B(T) : r \geqslant 0\}$ are independent and normally distributed with mean 0 and variance $s$. Furthermore, $B(s+t+T) - B(t+T) = \lim B(s+t+T_n) - B(t+T_n)$ is independent of $\mathscr{F}^+(T)$.

$\square$

# Reflection principle

**Theorem (Reflection principle)**

*If $T$ is a stopping and $\{B(t) : t \geqslant 0\}$ is standard Brownian motion, then the process $\{B^*(t) : t \geqslant 0\}$ called Brownian motion reflected at $T$ and defined by*

$$B^*(t) = B(t)\mathbf{1}_{t \leqslant T} + (2B(T) - B(t))\mathbf{1}_{t > T}$$

*is also a standard Brownian motion.*

# Reflection principle

**Proof.**

If $T$ is finite, by the strong Markov property

$$\{B(t+T) - B(T) : t \geqslant 0\}, \ \{-(B(t+T) - B(T)) : t \geqslant 0\}$$

are Brownian motions, and independent of $\{B(t) : 0 \leqslant t \leqslant T\}$. The process of glueing together paths is measurable, thus the two glueings induce the same distribution. □

# Maximum of Brownian motion

Let $B(t)$ be a one-dimensional Brownian motion. Define $M(t) = \max_{0 \leqslant s \leqslant t} B(s)$.

**Theorem**

*If* $a > 0$ *then* $\mathrm{Prob}_0(M(t) > a) = 2\,\mathrm{Prob}_0(B(t) > a) = \mathrm{Prob}_0(|B(t)| > a)$.

# Maximum of Brownian motion

**Proof.**

Let $T = \inf\{t \geqslant 0 : B(t) = a\}$ and let $\{B^*(t) : t \geqslant 0\}$ be Brownian motion reflected at stopping time $T$. Write

$$\{M(t) > a\} = \{B(t) > a\} \sqcup \{M(t) > a, B(t) \leqslant a\}.$$

The second event corresponds to $\{B^*(t) \geqslant a\}$, which has equal measure. □

# Convolution

## Definition

Given functions $f, g$, denote the *convolution* of functions $f$ and $g$ given by

$$f * g(x) := \int f(y)g(x-y)dy.$$

# Measure

Denote meas Lebesgue measure.

**Lemma**

*If $A_1, A_2 \subset \mathbb{R}^2$ are Borel sets of positive area, then*

$$\text{meas}(\{x \in \mathbb{R}^2 : \text{meas}(A_1 \cap (A_2 + x)) > 0\}) > 0.$$

# Measure

**Proof.**

Assume $A_1$ and $A_2$ are bounded. By Fubini

$$\begin{aligned}\int_{\mathbb{R}^2} \mathbf{1}_{A_1} * \mathbf{1}_{-A_2}(x)dx &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbf{1}_{A_1}(w)\mathbf{1}_{A_2}(w-x)dwdx \\ &= \int_{\mathbb{R}^2} \mathbf{1}_{A_1}(w)\left(\int_{\mathbb{R}^2} \mathbf{1}_{A_2}(w-x)dx\right)dw \\ &= \operatorname{meas}(A_1)\operatorname{meas}(A_2) > 0.\end{aligned}$$

Thus $\mathbf{1}_{A_1} * \mathbf{1}_{-A_2}(x) > 0$ on a set of positive measure. But $\mathbf{1}_{A_1} * \mathbf{1}_{-A_2}(x) = \operatorname{meas}(A_1 \cap (A_2 + x))$. $\square$

# Area of Brownian motion

**Theorem**

*Let $B[0,1]$ be a 2-d Brownian motion. Almost surely*

$$\mathrm{meas}(B[0,1]) = 0.$$

Proof.

- Let $X = \mathrm{meas}(B[0,1])$. We first check that $\mathrm{E}[X] < \infty$.
- In order that $X > a$ it is necessary that $B(t)$ leave the box of side length $\sqrt{a}$ surrounding the origin. Thus

$$\begin{aligned}\mathrm{Prob}(X > a) &\leqslant 2\,\mathrm{Prob}\left(\max_{t\in[0,1]} |W(t)| > \sqrt{a}/2\right)\\ &= 4\,\mathrm{Prob}(W(1) > \sqrt{a}/2) \leqslant 4e^{-a/8}\end{aligned}$$

  where $\{W(t) : t \geqslant 0\}$ is 1-d Brownian motion.
- As the estimate is integrable, $\mathrm{E}[X] < \infty$.

$\square$

# Area of Brownian motion

## Proof.

- Since $B(3t)$ and $\sqrt{3}B(t)$ have the same distribution,
$$\mathrm{E}[\mathrm{meas}(B[0,3])] = 3\,\mathrm{E}[X].$$
- Since
$$\mathrm{E}[\mathrm{meas}(B[0,3])] = \sum_{j=0}^{2} \mathrm{E}[\mathrm{meas}(B[j,j+1])]$$
it follows that, almost surely, the intersection of any two of the $B[j,j+1]$ has measure 0.
- Define Brownian motions $\{B_1(t) : t \in [0,1]\}$ and $\{B_2(t) : t \in [0,1]\}$ by $B_2(t) = B(t+2) - B(2) + B(1)$. These are independent of $Y = B(2) - B(1)$, (although not independent themselves).

$\square$

**Proof.**

- For $x \in \mathbb{R}^2$, let $R(x)$ be the area of $B_1[0,1] \cap (x + B_2[0,1])$. This is independent of $Y$.
- Calculate

$$0 = \mathrm{E}[\mathrm{meas}(B[0,1] \cap B[2,3])] = \mathrm{E}[R(Y)] = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\frac{|x|^2}{2}} \mathrm{E}[R(x)]dx.$$

- Thus, for a.e. $x$, $R(x) = 0$, so $\mathrm{meas}(B[0,1]) = \mathrm{meas}(B[2,3]) = 0$.

$\square$

# Area of Brownian motion

**Theorem**

*For any points* $x, y \in \mathbb{R}^d$*,* $d \geqslant 2$*, we have* $\mathrm{Prob}_x(y \in B(0,1]) = 0$.

# Area of Brownian motion

## Proof.

- It suffices to prove the result for $d = 2$ by projecting.
- By Fubini's theorem,

$$\int_{\mathbb{R}^2} \mathrm{Prob}_y(x \in B[0,1])dx = \mathrm{E}_y[\mathrm{meas}(B[0,1])] = 0.$$

- Hence, for a.e. $x$, $\mathrm{Prob}_y(x \in B[0,1]) = 0$.
- Thus

$$\begin{aligned}\mathrm{Prob}_y(x \in B[0,1]) &= \mathrm{Prob}_0(x - y \in B[0,1]) \\ &= \mathrm{Prob}_0(y - x \in B[0,1]) = \mathrm{Prob}_x(y \in B[0,1])\end{aligned}$$

and so $\mathrm{Prob}_x(y \in B[0,1]) = 0$ for a.e. $x$.

□

# Area of Brownian motion

**Proof.**

- Hence, for any $\epsilon > 0$, a.s. $\mathrm{Prob}_{B(\epsilon)}\{y \in B[0,1]\} = 0$.
- We obtain

$$\begin{aligned}\mathrm{Prob}_x(y \in B(0,1]) &= \lim_{\epsilon \downarrow 0} \mathrm{Prob}_x\{y \in B[\epsilon, 1]\} \\ &= \lim_{\epsilon \downarrow 0} \mathrm{E}_x \,\mathrm{Prob}_{B(\epsilon)}(y \in B[0, 1-\epsilon]) = 0.\end{aligned}$$

□

# Zeros

**Theorem**

*Let* $\{B(t) : t \geqslant 0\}$ *be a one dimensional Brownian motion and*

$$\mathrm{Zeros} = \{t \geqslant 0 : B(t) = 0\}$$

*its zero set. Almost surely,* $\mathrm{Zeros}$ *is a closed set with no isolated points.*

# Zeros

## Proof.

- $\mathrm{Zeros}$ is a.s. closed, since Brownian motion is a.s. continuous.
- For each rational $q \in [0, \infty)$ define

$$\tau_q = \inf\{t \geqslant q : B(t) = 0\}.$$

  Since the zero set is closed, this is a.s. a minimum.
- By the Strong Markov property, $\tau_q$ is not isolated from the right with probability 1, and this holds for all $q$ together.
- For those zeros $t$ not equal to $\tau_q$ for some $q$, let $q_n \uparrow t$ be a sequence of rationals. The points $\tau_{q_n}$ make $t$ not isolated from the left.

$\square$

# Markov processes

## Definition

A function $p : [0, \infty) \times \mathbb{R}^d \times \mathscr{B} \to \mathbb{R}$, where $\mathscr{B}$ is the Borel $\sigma$-algebra in $\mathbb{R}^d$ is a *Markov transition kernel* if

1. $p(\cdot, \cdot, A)$ is measurable as a function of $(t, x)$ for each $A \in \mathscr{B}$
2. $p(t, x, \cdot)$ is a Borel probability measure on $\mathbb{R}^d$ for all $t \geqslant 0$ and $x \in \mathbb{R}^d$, when integrating a function $f$ w.r.t. this measure we write
$$\int f(y)p(t, x, dy);$$
3. For all $A \in \mathscr{B}$, $x \in \mathbb{R}^d$ and $t, s > 0$,
$$p(t + s, x, A) = \int_{\mathbb{R}^d} p(t, y, A)p(s, x, dy).$$

# Markov processes

**Definition**

An adapted process $\{X(t) : t \geqslant 0\}$ is a *(time-homogeneous) Markov process* with transition kernel $p$ w.r.t. filtration $(\mathscr{F}(t) : t \geqslant 0)$ if, for all $t \geqslant s$ and Borel sets $A \in \mathscr{B}$ we have a.s.

$$\mathrm{Prob}(X(t) \in A | \mathscr{F}(s)) = p(t-s, X(s), A).$$

# Examples

**Example**

Brownian motion is a Markov process. The transition kernel $p$ has $p(t, x, \cdot)$ a normal distribution with mean $x$ and variance $t$.

**Example**

Reflected one-dimensional Brownian motion $\{X(t) : t \geqslant 0\}$ defined by $X(t) = |B(t)|$ is a Markov process. Its transition kernel $p(t, x, \cdot)$ is the law of $|Y|$ for $Y$ normally distributed with mean $x$ and variance $t$.

# The maximum of Brownian motion

**Theorem (Lévy, 1948)**

*Let $\{M(t) : t \geqslant 0\}$ be the maximum process of a 1d standard Brownian motion $\{B(t) : t \geqslant 0\}$, i.e.*

$$M(t) = \max_{0 \leqslant s \leqslant t} B(s).$$

*Then the process $\{Y(t) : t \geqslant 0\}$ defined by $Y(t) = M(t) - B(t)$ is a reflected Brownian motion.*

**Proof.**

- Fix $s \geqslant 0$ and consider the two processes $\{\hat{B}(t) : t \geqslant 0\}$ defined by
$$\hat{B}(t) = B(s+t) - B(s), \qquad t \geqslant 0,$$
and $\{\hat{M}(t) : t \geqslant 0\}$ defined by $\hat{M}(t) = \max_{0 \leqslant u \leqslant t} \hat{B}(u), t \geqslant 0$.
- We first check that, conditional on $\mathscr{F}^+(s)$, for $t \geqslant 0$, $Y(s+t)$ has the same distribution as $|Y(s) + \hat{B}(t)|$.
- This suffices for the theorem, since it implies that $\{Y(t) : t \geqslant 0\}$ is a Markov process with the transition kernel of reflected Brownian motion.

□

# The maximum of Brownian motion

**Proof.**

- Since

$$\begin{aligned} M(s+t) &= M(s) \vee (B(s) + \hat{M}(t)) \\ Y(s+t) &= (M(s) \vee (B(s) + \hat{M}(t))) - (B(s) + \hat{B}(t)) \\ &= (Y(s) \vee \hat{M}(t)) - \hat{B}(t). \end{aligned}$$

- It suffices to check, for every $y \geqslant 0$, $y \vee \hat{M}(t) - \hat{B}(t)$ has the same distribution as $|y + \hat{B}(t)|$.

□

# The maximum of Brownian motion

**Proof.**

- For any $a \geqslant 0$ write

$$P_1 = \mathrm{Prob}(y - \hat{B}(t) > a), \quad P_2 = \mathrm{Prob}(y - \hat{B}(t) \leqslant a, \hat{M}(t) - \hat{B}(t) > a)$$

so $\mathrm{Prob}(y \vee \hat{M}(t) - \hat{B}(t) > a) = P_1 + P_2$.

- By symmetry, $P_1 = \mathrm{Prob}(y + \hat{B}(t) > a)$, so it suffices to show that $P_2 = \mathrm{Prob}(y + \hat{B}(t) < -a)$.

$\square$

# The maximum of Brownian motion

## Proof.

- Define $W(u) := \hat{B}(t-u) - \hat{B}(t)$, $0 \leqslant u \leqslant t$, which is another Brownian motion.
- Define $M_W(t) = \max_{0 \leqslant u \leqslant t} W(u) = \hat{M}(t) - \hat{B}(t)$.
- Since $W(t) = -\hat{B}(t)$,

$$P_2 = \mathrm{Prob}(y + W(t) \leqslant a, M_W(t) > a).$$

- Let $W^*(u)$ be $W$ reflected at the first time that $W$ hits $a$. Thus

$$P_2 = \mathrm{Prob}(W^*(t) \geqslant a + y) = \mathrm{Prob}(y + \hat{B}(t) \leqslant -a).$$

Equality holds with probability 0, completing the proof.

□

# Stable subordinator

**Theorem**

*For any $a \geqslant 0$ define the stopping times*

$$T_a = \inf\{t \geqslant 0 : B(t) = a\}.$$

*Then $\{T_a : a \geqslant 0\}$ is an increasing Markov process with transition kernel given by the densities*

$$p(a,t,s) = \frac{a}{\sqrt{2\pi(s-t)^3}} \exp\left(-\frac{a^2}{2(s-t)}\right) \mathbf{1}(s > t), \qquad a > 0.$$

*This process is called the stable subordinator of index $\frac{1}{2}$.*

# Stable subordinator

**Proof.**

- Fix $a \geqslant b \geqslant 0$ and note that for all $t \geqslant 0$

$$\begin{aligned}
&\{T_a - T_b = t\} \\
&= \{B(T_b + s) - B(T_b) < a - b, \text{ for } s < t, \\
&\quad \text{and } B(T_b + t) - B(T_b) = a - b\}.
\end{aligned}$$

- By the strong Markov property, this is independent of $\mathscr{F}^+(T_b)$ and thus of $\{T_d : d \leqslant b\}$, which gives the Markov property of $\{T_a : a \geqslant 0\}$.

□

# Stable subordinator

**Proof.**

- Calculate

$$\begin{aligned}\mathrm{Prob}(T_a - T_b \leqslant t) = \mathrm{Prob}(T_{a-b} \leqslant t) &= \mathrm{Prob}\left(\max_{0\leqslant s\leqslant t} B(s) \geqslant a-b\right)\\ &= 2\,\mathrm{Prob}(B(t) \geqslant a-b) = 2\int_{a-b}^{\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}dx\\ &= (a-b)\int_0^t \frac{e^{-\frac{(a-b)^2}{2s}}}{\sqrt{2\pi s^3}}ds.\end{aligned}$$

□

# Cauchy process

**Theorem**

*Let $\{B(t) : t \geqslant 0\}$ be a planar Brownian motion, $B(t) = (B_1(t), B_2(t))$. Let*

$$V(a) = \{(x, y) \in \mathbb{R}^2 : x = a\}.$$

*Let $T(a)$ be the first hitting time of $V(a)$. The process $\{X(a) : a \geqslant 0\}$, $X(a) := B_2(T(a))$ is a Markov process with transition kernel*

$$p(a, x, A) = \frac{1}{\pi} \int_A \frac{a}{a^2 + (x - y)^2} dy.$$

*This process is called a Cauchy process.*

# Cauchy process

## Proof.

- The Markov property of $\{X(a) : a \geqslant 0\}$ follows from the strong Markov property of Brownian motion for $T(a)$.
- To calculate the transition density, recall that $T(a)$ has density

$$\frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{a^2}{2s}\right).$$

$T(a)$ is independent of $\{B_2(s) : s \geqslant 0\}$, so that $B_2(T(a))$ has density

$$\int_0^\infty \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right) \frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{a^2}{2s}\right) ds$$
$$= \int_0^\infty \frac{ae^{-\sigma}}{\pi(a^2+x^2)} d\sigma = \frac{a}{\pi(a^2+x^2)}.$$

$\square$

# Continuous time martingale

**Definition**

A real-valued stochastic process $\{X(t) : t \geqslant 0\}$ is a *martingale* w.r.t. a filtration $(\mathscr{F}(t) : t \geqslant 0)$ if it is adapted to the filtration, $\mathrm{E}[|X(t)|] < \infty$ for all $t \geqslant 0$ and, for any pair of times $0 \leqslant s \leqslant t$,

$$\mathrm{E}[X(t)|\mathscr{F}(s)] = X(s), \; a.s.$$

# Optional stopping theorem

**Theorem (Optional stopping theorem)**

*Suppose $\{X(t) : t \geqslant 0\}$ is a continuous martingale, and $0 \leqslant S \leqslant T$ are stopping times. If the process $\{X(t \wedge T) : t \geqslant 0\}$ is dominated by an integrable random variable $X$, i.e. $|X(t \wedge T)| \leqslant X$ a.s., for all $t \geqslant 0$, then*

$$\mathrm{E}[X(T)|\mathscr{F}(S)] = X(S), \; a.s.$$

This may be obtained from the discrete time result by discretization.

# Doob's maximal inequality

**Theorem (Doob's maximal inequality)**

*Suppose* $\{X(t) : t \geqslant 0\}$ *is a continuous martingale and* $p > 1$. *Then, for any* $t \geqslant 0$,

$$\mathrm{E}\left[\left(\sup_{0 \leqslant s \leqslant t} |X(s)|\right)^p\right] \leqslant \left(\frac{p}{p-1}\right)^p \mathrm{E}\left[|X(t)|^p\right].$$

Again, this can be proved from the corresponding result for discrete time martingales by discretization.

# Wald's lemma for Brownian motion

**Theorem (Wald's lemma for Brownian motion)**

*Let* $\{B(t) : t \geqslant 0\}$ *be a standard 1-d Brownian motion and* $T$ *a stopping time, such that either*

1. $\mathrm{E}[T] < \infty$
2. $\{B(t \wedge T) : t \geqslant 0\}$ *is dominated by an integrable random variable.*

*Then* $\mathrm{E}[B(T)] = 0$.

# Wald's lemma for Brownian motion

**Proof.**

Under the second condition one can apply the Optional stopping theorem with $S = 0$ to obtain $\mathrm{E}[B(T)] = 0$.
To reduce the first condition to the second, set

$$M_k = \max_{0 \leqslant t \leqslant 1} |B(t+k) - B(k)|, \qquad M = \sum_{k=1}^{\lceil T \rceil} M_k.$$

Notice $|B(t \wedge T)| \leqslant M$. We have

$$\mathrm{E}[M] = \mathrm{E}\left[\sum_{k=1}^{\lceil T \rceil} M_k\right] = \sum_{k=1}^{\infty} \mathrm{E}[\mathbf{1}(T > k-1) M_k]$$
$$= \sum_{k=1}^{\infty} \mathrm{Prob}(T > k-1)\, \mathrm{E}[M_k] = \mathrm{E}[M_0]\, \mathrm{E}[T+1] < \infty.$$

# Brownian motion in $L^2$

**Theorem**

*Let $S \leqslant T$ be stopping times and $\mathrm{E}[T] < \infty$. Then*

$$\mathrm{E}[(B(T))^2] = \mathrm{E}[(B(S))^2] + \mathrm{E}[(B(T) - B(S))^2].$$

**Proof.**

$$\begin{aligned}\mathrm{E}[B(T)^2] = \mathrm{E}[B(S)^2] &+ 2\,\mathrm{E}[B(S)\,\mathrm{E}[B(T) - B(S)|\mathscr{F}(S)]] \\ &+ \mathrm{E}[(B(T) - B(S))^2].\end{aligned}$$

The middle expectation vanishes. $\square$

# Brownian motion in $L^2$

**Theorem**

*Suppose* $\{B(t) : t \geqslant 0\}$ *is a 1-d Brownian motion. Then* $\{B(t)^2 - t : t \geqslant 0\}$ *is a martingale.*

**Proof.**

Calculate

$$\begin{aligned}
&\mathrm{E}[B(t)^2 - t|\mathscr{F}^+(s)] \\
&= \mathrm{E}[(B(t) - B(s))^2|\mathscr{F}^+(s)] + 2\,\mathrm{E}[B(t)B(s)|\mathscr{F}^+(s)] - B(s)^2 - t \\
&= B(s)^2 - s.
\end{aligned}$$

$\square$

# Wald's second lemma

**Theorem**

*Let $T$ be a stopping time for standard Brownian motion such that $\mathrm{E}[T] < \infty$. Then*

$$\mathrm{E}[B(T)^2] = \mathrm{E}[T].$$

# Wald's second lemma

## Proof.

- Define stopping time $T_n = \inf\{t \geqslant 0 : |B(t)| = n\}$
- Thus $\{B(t \wedge T \wedge T_n)^2 - t \wedge T \wedge T_n : t \geqslant 0\}$ is dominated by $n^2 + T$, which is integrable.
- By the optional stopping theorem, $\mathrm{E}[B(T \wedge T_n)^2] = \mathrm{E}[T \wedge T_n]$.
- Since $\mathrm{E}[B(T)^2] \geqslant \mathrm{E}[B(T \wedge T_n)^2]$,

$$\mathrm{E}[B(T)^2] \geqslant \lim_{n\to\infty} \mathrm{E}[B(T \wedge T_n)^2] = \lim_{n\to\infty} \mathrm{E}[T \wedge T_n] = \mathrm{E}[T].$$

- By Fatou,

$$\mathrm{E}[B(T)^2] \leqslant \liminf_{n\to\infty} \mathrm{E}[B(T \wedge T_n)^2] = \liminf_{n\to\infty} \mathrm{E}[T \wedge T_n] \leqslant \mathrm{E}[T].$$

$\square$

# Martingale properties of Brownian motion

Given twice differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ the Laplacian of $f$, written $\Delta f$, is

$$\Delta f(x) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}.$$

## Theorem

*Let $f : \mathbb{R}^d \to \mathbb{R}$ be twice continuously differentiable, and $\{B(t) : t \geqslant 0\}$ be a d-dimensional Brownian motion. Further suppose that, for all $t > 0$ and $x \in \mathbb{R}^d$, we have $\mathrm{E}_x[|f(B(t))|] < \infty$ and $\mathrm{E}_x[\int_0^t |\Delta f(B(s))| ds] < \infty$. Then the process $\{X(t) : t \geqslant 0\}$ defined by*

$$X(t) = f(B(t)) - \frac{1}{2} \int_0^t \Delta f(B(s)) ds$$

*is a martingale.*

# Martingale properties of Brownian motion

Proof.

For $0 \leqslant s < t$,

$$\mathrm{E}[X(t)|\mathscr{F}(s)]$$
$$= \mathrm{E}_{B(s)}[f(B(t-s))] - \frac{1}{2}\int_0^s \Delta f(B(u))du - \int_0^{t-s} \mathrm{E}_{B(s)}[\frac{1}{2}\Delta f(B(u))]du.$$

The Markov transition kernel of Brownian motion satisfies $\frac{1}{2}\Delta p(t,x,y) = \frac{\partial}{\partial t}p(t,x,y)$, so that, integrating by parts,

$$\mathrm{E}_{B(s)}[\frac{1}{2}\Delta f(B(u))] = \frac{1}{2}\int p(u, B(s), x)\Delta f(x)dx$$
$$= \frac{1}{2}\int \Delta p(u, B(s), x) f(x)dx = \int \frac{\partial}{\partial u} p(u, B(s), x) f(x)dx$$

$\square$

# Martingale properties of Brownian motion

**Proof.**

Thus

$$\int_0^{t-s} \mathrm{E}_{B(s)}[\frac{1}{2}\Delta f(B(u))]du = \lim_{\epsilon \downarrow 0} \int \left[\int_\epsilon^{t-s} \frac{\partial}{\partial u} p(u, B(s), x) du\right] f(x) dx$$
$$= \int p(t-s, B(s), x) f(x) dx - \lim_{\epsilon \downarrow 0} \int p(\epsilon, B(s), x) f(x) dx$$
$$= \mathrm{E}_{B(s)}[f(B(t-s))] - f(B(s))$$

which proves that $X$ is a martingale. □