Math 639: Lecture 16

Equidistribution on nilmanifolds

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Quantitative equidistribution on nilmanifolds

This lecture is based on the paper "The quantitative behavior of polynomial orbits on nilmanifolds" by B. Green and T. Tao, *Annals of Math* 175 (2012), 465–540.

Filtrations, nilmanifolds

Definition (Filtrations and nilmanifolds)

- Let G be a connected, simply connected nilpotent Lie group with identity 1_G. A *filtration* G_* on G is a sequence of closed connected subgroups

$$G = G_0 = G_1 \supset G_2 \supset \cdots \supset G_d \supset G_{d+1} = \{1_G\}$$

  which has the property that $[G_i, G_j] \subset G_{i+j}$ for all integers $i, j \geqslant 0$.
- Let $\Gamma \subset G$ be a uniform (discrete, cocompact) subgroup. The quotient $G/\Gamma = \{g\Gamma : g \in G\}$ is called a *nilmanifold*.

We write $m = \dim G$ and $m_i = \dim G_i$.

Filtrations, nilmanifolds

In a nilpotent Lie group, the *lower central series*, defined by

$$G = G_0 = G_1, \; G_{i+1} = [G, G_i]$$

and terminating with $G_{s+1} = \{id_G\}$, is an example of a filtration. The number s is called the *step* of G. In general, in a filtration with $G_{d+1} = \{1_G\}$, d is called the *degree*.

Examples

Example

- In the case $s = 1$, up to a linear transformation the nilmanifolds are given by tori, $G = \mathbb{R}^m$, $\Gamma = \mathbb{Z}^m$. The lcs filtration is $G = G_0 = G_1$, $G_2 = \{1_G\}$.

Examples

Example

- The *Heisenberg nilmanifold* has $s = 2$,

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}.$$

The lcs filtration has $G = G_0 = G_1$, $G_2 = \begin{pmatrix} 1 & 0 & \mathbb{R} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $G_3 = \{1_G\}$.

A fundamental domain for G/Γ is

$$\left\{ \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} : 0 \leqslant x_1, x_2, x_3 < 1 \right\}.$$

Mal'cev bases

Definition (Mal'cev bases)

Let G/Γ be a m-dimensional nilmanifold and let G_* be a filtration. A basis $\mathscr{X} = \{X_1, ..., X_m\}$ for the Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is called a *Mal'cev basis* for G/Γ adapted to G_* if the following four conditions are satisfied.

1. For each $j = 0, ..., m-1$ the subspace $\mathfrak{h}_j := \mathrm{Span}\{X_{j+1}, ..., X_m\}$ is a Lie algebra ideal in $\mathfrak{g}$, and $H_j = \exp \mathfrak{h}_j$ is a normal Lie subgroup of G.
2. For every $0 \leqslant i \leqslant s$, we have $G_i = H_{m-m_i}$
3. Each $g \in G$ has a unique expression as $\exp(t_1 X_1) \cdots \exp(t_m X_m)$ for $t_i \in \mathbb{R}$
4. Γ consists of those points with all $t_i \in \mathbb{Z}$.

Mal'cev bases

- Mal'cev proved that any nilmanifold G/Γ can be equipped with a Mal'cev basis adapted to the lower central series filtration.
- Given a Mal'cev basis, the coordinates t_i are referred to as *Mal'cev coordinates* and the *Mal'cev coordinate map* $\psi_{\mathscr{X}} : G \to \mathbb{R}^m$ is the map

$$\psi_{\mathscr{X}}(g) := (t_1, ..., t_m).$$

Mal'cev metric

Definition

Let G/Γ be a nilmanifold with Mal'cev basis $\mathscr{X}$. We define $d = d_{\mathscr{X}} : G \times G \to \mathbb{R}_{\geqslant 0}$ to be the largest metric such that for all $x, y \in G$,

$$d(x, y) \leqslant \|\psi(xy^{-1})\|_\infty.$$

Explicitly,

$$d(x, y) =$$
$$\inf \left\{ \sum_{i=1}^n \min(\|\psi(x_{i-1}x_i^{-1})\|_\infty, \|\psi(x_i x_{i-1}^{-1})\|_\infty) : x_0, \ldots, x_n \in G; x_0 = x; x_n = y \right\}$$

and

$$d(x\Gamma, y\Gamma) = \inf_{\gamma \in \Gamma} d(x, y\gamma).$$

Rationality of a Mal'cev basis

Definition (Height)

The *(naive) height* of a real number x is defined to be $\max(|a|, |b|)$ if $x = \frac{a}{b}$ is rational in reduced form, and ∞ if x is irrational.

Definition (Rationality of a basis)

Let G/Γ be a nilmanifold, and let $Q > 0$. We say that a Mal'cev basis $\mathscr{X}$ for G/Γ is *Q-rational* if all of the structure constants c_{ijk} in the relations

$$[X_i, X_j] = \sum_k c_{ijk} X_k$$

are rational with height at most Q.

Equidistribution

Definition (Equidistribution)

Let G/Γ be a nilmanifold endowed with a unique probability Haar measure dx.

- An infinite sequence $\{g(n)\Gamma\}_{n\in\mathbb{N}}$ is *equidistributed* if for all continuous functions $F : G/\Gamma \to \mathbb{C}$,

$$\lim_{N\to\infty} \mathrm{E}_{n\in[N]}[F(g(n)\Gamma)] = \int_{G/\Gamma} F(x)dx.$$

- An infinite sequence $\{g(n)\Gamma\}_{n\in\mathbb{Z}}$ in G/Γ is *totally equidistributed* if the sequences $\{g(an+r)\Gamma\}_{n\in\mathbb{N}}$ are equidistributed for all $a \in \mathbb{Z}\backslash\{0\}$ and all $r \in \mathbb{Z}$.

Lipschitz functions

Definition (Lipschitz functions)

Let (G/Γ) be a nilmanifold with Mal'cev basis $\mathscr{X}$. The *Lipschitz norm* of a function $F : G/\Gamma \to \mathbb{C}$ is

$$\|F\|_{\mathrm{Lip}} := \|F\|_\infty + \sup_{x \neq y \in G/\Gamma} \frac{|F(x) - F(y)|}{d(x,y)}.$$

A function F is said to be *Lipschitz* if it has finite Lipschitz norm.

Quantitative equidistribution

Definition (Quantitative equidistribution)

Let $(G/\Gamma, dx)$ as above with a Mal'cev basis $\mathscr{X}$, and let be given an error tolerance $\delta > 0$ and a length N.

- A finite sequence $\{g(n)\Gamma\}_{n\in[N]}$ is said to be δ-*equidistributed* if

$$\left|\mathbf{E}_{n\in[N]}[F(g(n)\Gamma)] - \int_{G/\Gamma} F(x)dx\right| \leqslant \delta\|F\|_{\mathrm{Lip}}$$

  for all Lipschitz functions $F : G/\Gamma \to \mathbb{C}$.

- The sequence is *totally* δ-*equidistributed* if

$$\left|\mathbf{E}_{n\in P}[F(g(n)\Gamma)] - \int_{G/\Gamma} F(x)dx\right| \leqslant \delta\|F\|_{\mathrm{Lip}}$$

  holds for all arithmetic progressions $P \subset [N]$ of length at least δN.

Linear sequences

Definition (Linear sequences)

A *linear sequence* in a group G is any sequence $g : \mathbb{Z} \to G$ of the form $g(n) := a^n x$ for some $a, x \in G$. A *linear sequence* in a nilmanifold G/Γ is a sequence of the form $\{g(n)\Gamma\}_{n \in \mathbb{Z}}$, where $g : \mathbb{Z} \to G$ is a linear sequence in G.

In the case $G = \mathbb{R}^m$, $\Gamma = \mathbb{Z}^m$, a linear sequence takes the form $an + x \bmod \mathbb{Z}^m$.

Qualitative Kronecker

Theorem (Qualitative Kronecker theorem)

Let $m \geqslant 1$*, and let* $(g(n) \bmod \mathbb{Z}^m)_{n\in\mathbb{N}}$ *be a linear sequence in the torus* $\mathbb{R}^m/\mathbb{Z}^m$*. Then exactly one of the following statements is true:*

1. $(g(n) \bmod \mathbb{Z}^m)$ *is equidistributed in* $\mathbb{R}^m/\mathbb{Z}^m$
2. *There exists a non-trivial character* $\eta : \mathbb{R}^m \to \mathbb{R}/\mathbb{Z}$ *which annihilates* $\mathbb{Z}^m$ *but does not vanish entirely, such that* $\eta \circ g$ *is constant.*

Horizontal torus

Definition (Horizontal torus)

- Given a nilmanifold G/Γ, the *horizontal torus* is defined to be $(G/\Gamma)_{\text{ab}} := G/[G,G]\Gamma$. This torus is isomorphic to $\mathbb{R}^{m_{\text{ab}}}/\mathbb{Z}^{m_{\text{ab}}}$ where $m_{\text{ab}} = \dim(G) - \dim([G,G])$.
- A *horizontal character* is a continuous homomorphism $\eta : G \to \mathbb{R}/\mathbb{Z}$ which annihilates Γ. Note that η in fact annihilates $[G,G]\Gamma$.
- A horizontal character is *non-trivial* if it does not vanish identically.
- Given a Mal'cev basis $\mathscr{X}$ for G/Γ, a horizontal character η may be written $\eta(g) = k \cdot \psi(g)$ for some unique $k \in \mathbb{Z}^m$. The *norm* of η is $\|\eta\| := \|k\|$.

The Heisenberg nilmanifold

Example

Let G/Γ be the Heisenberg nilmanifold. Then $[G, G] = \begin{pmatrix} 1 & 0 & \mathbb{R} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $(G/\Gamma)_{\mathrm{ab}}$ may be identified with $\mathbb{R}^2/\mathbb{Z}^2$, with projection map π given by

$$\pi\left[\begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}\right] := (x_1, x_3).$$

Leon Green's theorem

Leon Green proved the following generalization to Kronecker's theorem in the case of a linear orbit in a nilmanifold.

Theorem (Leon Green's theorem)

Let $\{g(n)\Gamma\}_{n\in\mathbb{Z}}$ be a linear sequence in a nilmanifold G/Γ. Then exactly one of the following statements is true:

1. $\{g(n)\Gamma\}_{n\in\mathbb{N}}$ *is equidistributed in* G/Γ
2. *There is a nontrivial horizontal character* $\eta : G/\Gamma \to \mathbb{R}/\mathbb{Z}$ *such that* $\eta \circ g$ *is a constant.*

In other words, a linear sequence is equidistributed in a nilmanifold if and only if the sequence projected to the abelianization is equidistributed there.

Leon Green's theorem

Example

Write $\{x\} = x - \lfloor x \rfloor$. The linear sequence

$$g_n = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n\alpha & \binom{n}{2}\alpha\beta \\ 0 & 1 & n\beta \\ 0 & 0 & 1 \end{pmatrix}$$

has reduction into a fundamental domain for the Heisenberg nilmanifold G/Γ given by

$$g_n\Gamma = \begin{pmatrix} 1 & \{n\alpha\} & \left\{\binom{n}{2}\alpha\beta - n\alpha\lfloor n\beta \rfloor\right\} \\ 0 & 1 & \{n\beta\} \\ 0 & 0 & 1 \end{pmatrix}\Gamma.$$

By Leon Green's theorem, this sequence is equidistributed in G/Γ if and only if $(1, \alpha, \beta)$ are linearly independent over $\mathbb{Q}$.

Polynomial sequences

Definition (Polynomial sequences)

- Let G be a nilpotent group with filtration G_*. Let $g : \mathbb{Z} \to G$ be a sequence. If $h \in \mathbb{Z}$, write $D_h g(n) := g(n+h)g(n)^{-1}$. (This differs slightly from last lecture.)
- We say that g is a *polynomial sequence with coefficients in* G_*, and write $g \in \mathrm{poly}(\mathbb{Z}, G_*)$, if $D_{h_i} \cdots D_{h_1} g$ takes values in G_i for all $i \in \mathbb{Z}_{>0}$ and all $h_1, ..., h_i \in \mathbb{Z}$.

Leibman proved the following generalization of Leon Green's theorem.

Theorem (Leibman's theorem)

Suppose that G/Γ *is a nilmanifold and that* $g : \mathbb{Z} \to G$ *is a polynomial sequence. Then exactly one of the following statements is true:*

1. $\{g(n)\Gamma\}_{n\in\mathbb{N}}$ *is equidistributed in* G/Γ.
2. *There exists a nontrivial horizontal character* $\eta : G \to \mathbb{R}/\mathbb{Z}$ *such that* $\eta \circ g$ *is constant.*

Quantitative Leibman's theorem

Green and Tao prove the following quantitative form of Leibman's theorem.

Theorem (Quantitative Leibman theorem)

Let G/Γ be an m-dimensional nilmanifold with filtration G_* of degree d and a $\frac{1}{\delta}$-rational Mal'cev basis $\mathscr{X}$. Suppose that $g \in \mathrm{poly}(\mathbb{Z}, G_*)$. Then at least one of the following statements is true.

1. $\{g(n)\Gamma\}_{n\in[N]}$ *is δ-equidistributed in G/Γ*
2. *There is a non-trivial horizontal character $\eta : G/\Gamma \to \mathbb{R}/\mathbb{Z}$ with $\|\eta\| \ll \delta^{-O_{m,d}(1)}$ such that*

$$\|\eta \circ g(n) - \eta \circ g(n-1)\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{\delta^{-O_{m,d}(1)}}{N}$$

   for all $n \in \{1, \ldots, N\}$.

For $x \in \mathbb{R}$, $\|x\|_{\mathbb{R}/\mathbb{Z}}$ is the distance to the nearest integer.

Example

Example

Let $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$ and $g(n) = (\frac{1}{2} + \sigma)n$ where $0 < \sigma \leqslant \frac{\delta}{100}$ is a parameter.

- If N is much larger than $\frac{1}{\sigma}$ then $\{g(n) \bmod \mathbb{Z}\}_{n \in [N]}$ is δ-equidistributed.
- If N is much smaller than $\frac{1}{\sigma}$ then $\{g(n) \bmod \mathbb{Z}\}_{n \in [N]}$ fails to be equidistributed as it is concentrated near 0 and $\frac{1}{2}$. The character $\eta(x) = 2x \bmod 1$ satisfies

$$\|\eta \circ g(n) - \eta \circ g(n-1)\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{1}{N}.$$

Rational sequences

Definition

- Let G/Γ be a nilmanifold, and let $Q > 0$ be a parameter. We say that $\gamma \in G$ is *Q-rational* if $\gamma^r \in \Gamma$ for some integer r, $0 < r \leqslant Q$.
- A *Q-rational point* is any point in G/Γ of the form $\gamma\Gamma$ for some Q-rational group element γ.
- A sequence $\{\gamma(n)\}_{n \in \mathbb{Z}}$ is *Q-rational* if every element $\gamma(n)\Gamma$ in the sequence is a Q-rational point.

Rational subgroups

Definition

Let $Q > 0$.

- Suppose a nilmanifold G/Γ is given with Mal'cev basis $\mathscr{X} = \{X_1, ..., X_m\}$.
- Suppose that $G' \subset G$ is a closed connected subgroup.

We say that G' is *Q-rational* relative to $\mathscr{X}$ if the Lie algebra $\mathfrak{g}'$ has a basis $\mathscr{X}' = \{X_1', ..., X_{m'}'\}$ consisting of linear combinations $\sum_{i=1}^m a_i X_i$ where the a_i are rational numbers of height at most Q.

Smooth sequences

Definition

Let G/Γ be a nilmanifold with a Mal'cev basis $\mathscr{X}$, and let $M, N \geqslant 1$. We say that the sequence $\{\epsilon(n)\}_{n \in \mathbb{Z}}$ in G is (M, N)-*smooth* if we have

$$d(\epsilon(n), 1_G) \leqslant M$$

and

$$d(\epsilon(n), \epsilon(n-1)) \leqslant \frac{M}{N}$$

for all $n \in [N]$.

Factorization theorem

Green and Tao prove the following factorization theorem as part of their program giving an asymptotic for the number of k-term arithmetic progressions in the prime numbers less than X.

Factorization theorem

Theorem (Factorization theorem)

- *Let $m, d \geqslant 0$, and let $M_0, N \geqslant 1$ and $A > 0$ be real numbers.*
- *Suppose that G/Γ is an m-dimensional nilmanifold with a filtration G_* of degree d*
- *Suppose that $\mathscr{X}$ is an M_0-rational Mal'cev basis adapted to G_* and that $g \in \mathrm{poly}(\mathbb{Z}, G_*)$.*

Then there is an integer M with $M_0 \leqslant M \ll M_0^{O_{A,m,d}(1)}$, a rational subgroup $G' \subset G$, a Mal'cev basis $\mathscr{X}'$ for $G'/(G' \cap \Gamma)$, and a decomposition $g = \epsilon g' \gamma$ into sequences in $\mathrm{poly}(\mathbb{Z}, G_*)$ satisfying

1. $\epsilon : \mathbb{Z} \to G$ *is (M, N)-smooth*
2. $g' : \mathbb{Z} \to G'$ *satisfies $\{g'(n)\Gamma'\}_{n \in [N]}$ is totally $1/M^A$-equidistributed*
3. $\gamma : \mathbb{Z} \to G$ *is M-rational, and $\{\gamma(n)\Gamma\}_{n \in \mathbb{Z}}$ is periodic with period at most M.*

Quantitative Kronecker Theorem

As a warm-up we prove the following quantitative Kronecker Theorem.

Theorem (Quantitative Kronecker Theorem)

Let $m \geqslant 1$, let $0 < \delta < \frac{1}{2}$, and let $\alpha \in \mathbb{R}^m$. If the sequence $\{\alpha n \bmod \mathbb{Z}^m\}_{n \in [N]}$ is not δ-equidistributed in the additive torus $\mathbb{R}^m/\mathbb{Z}^m$, then there exists $k \in \mathbb{Z}^m$ with $0 < |k| \ll \delta^{-O_m(1)}$ such that $\|k \cdot \alpha\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{\delta^{-O_m(1)}}{N}$.

Quantitative Kronecker Theorem

Proof.

- Suppose that $\{\alpha n \bmod \mathbb{Z}^m\}_{n\in[N]}$ is not δ-equidistributed. Thus there exists a Lipschitz function $F : \mathbb{R}^m/\mathbb{Z}^m \to \mathbb{R}$, $\|F\|_{\mathrm{Lip}} = 1$

$$\left| \mathbf{E}_{n\in[N]}[F(\alpha n)] - \int_{\mathbb{R}^m/\mathbb{Z}^m} F d\theta \right| > \delta.$$

After replacing δ with $\delta/2$ and translating F by a constant and rescaling, we can assume $\int F = 0$. Also, we may assume that F is smooth.

□

Quantitative Kronecker Theorem

Proof.

- Fejér kernel $K : \mathbb{R}^m/\mathbb{Z}^m \to \mathbb{R}^+$, $K(\theta) := \frac{1_Q}{\text{meas}(Q)} * \frac{1_Q}{\text{meas}(Q)}(\theta)$, where $Q := \left[-\frac{\delta}{16m}, \frac{\delta}{16m}\right]^m \subset \mathbb{R}^m/\mathbb{Z}^m$ has F.T., for $k \in \mathbb{Z}^m$,

$$\hat{K}(k) = \prod_{i=1}^m \left(\frac{\sin \frac{\pi k_i \delta}{8m}}{\frac{\pi k_i \delta}{8m}} \right)^2,$$

  where the ratio is interpretted as 1 where $k_i = 0$.

- Bounding the numerator by 1, for $M \geqslant 1$,

$$\sum_{k \in \mathbb{Z}^m, \|k\|_2 > M} |\hat{K}(k)| \ll_m \delta^{-2m} M^{-1}.$$

□

Quantitative Kronecker Theorem

Proof.

- Bound $|\hat{F}(k)| \leqslant \|F\|_\infty \leqslant \|F\|_{\mathrm{Lip}} \leqslant 1$.
- Set $F_1 := F * K$. Since $\|F\|_{\mathrm{Lip}} = 1$ and K is supported in Q,

$$\|F - F_1\|_\infty \leqslant \frac{\delta}{8}.$$

- Choose $M := C_m \delta^{-2m-1}$ for m sufficiently large, and set

$$F_2(\theta) := \sum_{k \in \mathbb{Z}^m : 0 < \|k\|_2 \leqslant M} \hat{F}_1(k) e(k \cdot \theta).$$

  Thus $\|F_1 - F_2\|_\infty \leqslant \frac{\delta}{8}$.

□

Quantitative Kronecker Theorem

Proof.

- We've arranged that

$$\left|\mathrm{E}_{n\in[N]}[F_2(n\alpha)]\right| \geqslant \frac{\delta}{4}.$$

Thus there exists some k, $0 < |k| \leqslant M$ such that

$$\left|\mathrm{E}_{n\in[N]}[e(nk\cdot\alpha)]\right| \gg_m \delta M^{-m} \gg \delta^{O_m(1)}.$$

- The geometric series bound $\left|\mathrm{E}_{n\in[N]}[e(nt)]\right| \ll \min\left(1, \frac{1}{N\|t\|_{\mathbb{R}/\mathbb{Z}}}\right)$ implies

$$\|k\cdot\alpha\|_{\mathbb{R}/\mathbb{Z}} \ll_m \frac{\delta^{-O_m(1)}}{N}.$$

□

Strongly recurrent sequences

The following one-dimensional version of Kronecker's theorem gives extra information in the case that a small interval is hit often.

Lemma

Let $\alpha \in \mathbb{R}$, $0 < \delta < \frac{1}{2}$, $0 < \epsilon \leqslant \frac{\delta}{2}$, and let $I \subset \mathbb{R}/\mathbb{Z}$ be an interval of length ϵ. If $\alpha n \in I$ for at least δN values of $n \in [N]$ then there is some $k \in \mathbb{Z}$ with $0 < |k| \ll \delta^{-O(1)}$ such that

$$\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{\epsilon\delta^{-O(1)}}{N}.$$

Strongly recurrent sequences

Proof.

- By choosing a function F which is a piecewise linear approximation to I one can check that $\{\alpha n \bmod 1\}_{n\in[N]}$ is not $\frac{\delta^2}{10}$ equidistributed.
- Choose $0 \neq k \in \mathbb{Z}$ such that $|k| \ll \delta^{-O(1)}$ and $\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{\delta^{-O(1)}}{N}$.
- Let $\beta = \|k\alpha\|_{\mathbb{R}/\mathbb{Z}}$, and assume $\beta \neq 0$, since otherwise we're done.

□

Strongly recurrent sequences

Proof.

- As n' ranges over an interval of integers J of length at most $\frac{1}{\beta}$, the numbers $\alpha(n_0 + qn')\mathbb{Z}$ are increasing through a fundamental domain for $\mathbb{R}/\mathbb{Z}$, and thus the number that land in I is at most $1 + \frac{\epsilon}{\beta}$.
- Divide $[N]$ into at most $2k + \beta N$ progressions of form $\{n_0 + kn' : n' \in J\}$ to obtain

$$\delta N \leqslant \#\{n \in [N] : \alpha n \bmod 1 \in I\} \leqslant \left(1 + \frac{\epsilon}{\beta}\right)(2k + \beta N) \ll k + \frac{\epsilon k}{\beta} + \beta N + \epsilon N.$$

- We can assume that $N \geqslant \delta^{-O(1)}$ and that $\epsilon < \delta^{O(1)}$. The only term that is relevant is $\delta N \ll \frac{k\epsilon}{\beta}$, which gives the claim.

□

Vertical torus

Definition

Let G/Γ be a nilmanifold and G_* a filtration of degree d. Then G_d is in the center of G.

- The *vertical torus* is $G_d/(\Gamma \cap G_d)$.
- The *vertical dimension* is $m_d = \dim G_d$
- A *vertical character* is a continuous homomorphism $\xi : G_d/(\Gamma \cap G_d) \to \mathbb{R}/\mathbb{Z}$. Such a character has the form $\xi(x) = k \cdot x$, $k \in \mathbb{Z}^d$, by identifying G_d with the last m_d Mal'cev coordinates.
- Let $F : G/\Gamma \to \mathbb{C}$ be a Lipschitz function and let ξ be a vertical character. F has *vertical oscillation* ξ if

$$F(g_d \cdot x) = e(\xi(g_d))F(x), \qquad g_d \in G_d, x \in G/\Gamma.$$

Equidistribution in subspaces

Definition

Let $g : \mathbb{Z} \to G$ be a polynomial sequence. We say that $\{g(n)\Gamma\}_{n\in[N]}$ is δ-equidistributed along a vertical character ξ if

$$\left| \mathbf{E}_{n\in[N]}[F(g(n)\Gamma)] - \int_{G/\Gamma} F(x)dx \right| \leqslant \delta \|F\|_{\mathrm{Lip}}$$

for all Lipschitz functions $F : G/\Gamma \to \mathbb{C}$ with vertical oscillation ξ.

Vertical oscillation reduction

Lemma

- *Let G/Γ be a nilmanifold with filtration G_* of degree d.*
- *Let m_d be the vertical dimension, and let $0 < \delta \leqslant \frac{1}{2}$.*
- *Suppose that $g : \mathbb{Z} \to G$ is a polynomial sequence and that $\{g(n)\Gamma\}_{n \in [N]}$ is not δ-equidistributed.*

Then there is a vertical character ξ with $|\xi| \ll \delta^{-O_{m_d}(1)}$ such that $\{g(n)\Gamma\}_{n \in [N]}$ is not $\delta^{O_{m_d}(1)}$-equidistributed along the vertical frequency ξ.

Vertical oscillation reduction

Proof.

This follows as in the quantitative Kronecker theorem.

- Replacing δ with $\frac{\delta}{2}$, assume $\int_{G/\Gamma} F = 0$, $\|F\|_{\mathrm{Lip}} = 1$ and F is smooth.
- Let K be the m_d dimension Fejér kernel. Convolve with K in $G_d/(\Gamma \cap G_d)$ fibers to obtain

$$F_1(y) := \int_{\mathbb{R}^{m_d}/\mathbb{Z}^{m_d}} F(\theta y) K(\theta) d\theta$$

- Write $F_1(y) = \sum_{k \in \mathbb{Z}^d} F^\wedge(y;k)\hat{K}(k)$ and $(Q = C_{m_d}\delta^{-2m_d-1})$

$$F_2(y) := \sum_{k \in \mathbb{Z}^{m_d} : \|k\| \leqslant Q} F^\wedge(y;k)\hat{K}(k)$$

Since $\|F - F_2\|_\infty \leqslant \frac{\delta}{4}$, the argument goes through as before.

□

Theorem (van der Corput's inequality)

Let N, H be positive integers and suppose that $\{a_n\}_{n\in[N]}$ is a sequence of complex numbers. Extend $\{a_n\}$ to all of $\mathbb{Z}$ by defining $a_n := 0$ for $n \notin [N]$.

$$\left|\mathbf{E}_{n\in[N]}[a_n]\right|^2 \leqslant \frac{N+H}{HN} \sum_{|h|\leqslant H} \left(1 - \frac{|h|}{H}\right) \mathbf{E}_{n\in[N]}[a_n \overline{a_{n+h}}].$$

In the classical theory of oscillating sums $\sum_n e(P(n))$, van der Corput's inequality is used to reduce the degree of the polynomial P.

Proof of van der Corput's inequality.

Write $\sum_n a_n = \frac{1}{H}\sum_{-H<n\leqslant N}\sum_{h=0}^{H-1} a_{n+h}$. By Cauchy-Schwarz,

$$\begin{aligned}\left|\sum_n a_n\right|^2 &= \frac{1}{H^2}\left|\sum_{-H<n\leqslant N}\sum_{h=0}^{H-1} a_{n+h}\right|^2 \\ &\leqslant \frac{N+H}{H^2}\sum_{-H<n\leqslant N}\left|\sum_{h=0}^{H-1} a_{n+h}\right|^2 \\ &= \frac{N+H}{H^2}\sum_{-H<n\leqslant N}\sum_{h,h'=0}^{H-1} a_{n+h}\overline{a_{n+h'}}.\end{aligned}$$

This rearranges to the claimed inequality. □

van der Corput's inequality

Theorem

Let N *be a positive integer, and suppose that* $\{a_n\}_{n\in[N]}$ *is a sequence of complex numbers with* $|a_n| \leqslant 1$*. Extend* $\{a_n\}$ *to all of* $\mathbb{Z}$ *by defining* $a_n := 0$ *when* $n \notin [N]$*. Suppose that* $0 < \delta < 1$ *and that*

$$\left|\mathrm{E}_{n\in[N]}[a_n]\right| \geqslant \delta.$$

Then for at least $\frac{\delta^2 N}{8}$ *values of* $h \in [N]$*, we have*

$$\left|\mathrm{E}_{n\in[N]}[a_{n+h}\overline{a_n}]\right| \geqslant \frac{\delta^2}{8}.$$

van der Corput's inequality

Proof.

We can assume that $N \leqslant \frac{4}{\delta^2}$. The proof is by contradiction. Choose $H = N$ in the previous theorem to obtain

$$\begin{aligned}\delta^2 \leqslant \left|\mathbf{E}_{n\in[N]}\, a_n\right|^2 &\leqslant \frac{2}{N}\sum_{|h|\leqslant N}\left|\mathbf{E}_{n\in[N]}\left[a_n\overline{a_{n+h}}\right]\right| \\ &\leqslant \frac{2}{N}\left(1 + 2\left(\frac{\delta^2 N}{8} + \frac{\delta^2 N}{8}\right)\right).\end{aligned}$$

Rearranging produces the inequality. □

The Heisenberg nilmanifold

The Heisenberg group has Lie algebra $\mathfrak{g} = \begin{pmatrix} 0 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & \mathbb{R} \\ 0 & 0 & 0 \end{pmatrix}$. The exponential map is given by

$$\exp \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & y + \frac{xz}{2} \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

$$\log \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x & y - \frac{xz}{2} \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}.$$

The Heisenberg nilmanifold

Let

$$X_1 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_3 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\mathscr{X} = \{X_1, X_2, X_3\}$ is a Mal'cev basis adapted to the lcs filtration G_*,

$$\exp(t_1 X_1) \exp(t_2 X_2) \exp(t_3 X_3) = \begin{pmatrix} 1 & t_1 & t_1 t_2 + t_3 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Mal'cev coordinate map is

$$\psi \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = (x, z, y - xz).$$

Projection onto the horizontal torus is given by $\pi \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = (x, z).$

Heisenberg case

We prove the main theorem in the special case of a linear sequence in the Heisenberg nilmanifold. This already illustrates many of the essential ingredients of the more general proof.

Theorem

- *Let G/Γ be the Heisenberg nilmanifold with Mal'cev basis given, and let $g : \mathbb{Z} \to G$ be a linear sequence $g(n) = a^n$.*
- *Let $\delta > 0$ be a parameter and let $N \geqslant 1$ be an integer.*

Then either $\{g(n)\Gamma\}_{n \in [N]}$ is δ-equidistributed, or else there is a horizontal character η with $0 < |\eta| \ll \delta^{-O(1)}$ such that $\|\eta(a)\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{\delta^{-O(1)}}{N}$.

Heisenberg case

Proof.

- Assume that the sequence is not δ-equidistributed. Applying the vertical frequency decomposition, there exists $F : G/\Gamma \to \mathbb{C}$, $\|F\|_{\mathrm{Lip}} = 1$ of vertical frequency ξ with $\|\xi\| \ll \delta^{-O(1)}$, such that

$$\left| \mathrm{E}_{n\in[N]}[F(a^n\Gamma)] - \int_{G/\Gamma} F(x)dx \right| \gg \delta^{O(1)}.$$

- If $\xi \equiv 0$ then F is G_2-invariant, so there exists $\tilde{F} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}$ such that $F(x) = \tilde{F}(\pi(x))$. One has $\|\tilde{F}\|_{\mathrm{Lip}} \leqslant 1$ and

$$\left| \mathrm{E}_{n\in[N]}\, \tilde{F}(n\pi(a)) - \int_{\mathbb{R}^2/\mathbb{Z}^2} \tilde{F}(x)dx \right| \gg \delta^{O(1)} \|\tilde{F}\|_{\mathrm{Lip}}.$$

The claim now follows from Kronecker's theorem.

□

Heisenberg case

Proof.

- If $\xi \not\equiv 0$ then F is mean zero, by integrating in the G_2 direction. Hence
$$\left|\mathrm{E}_{n\in[N]}[F(a^n\Gamma)]\right| \geqslant \delta^{O(1)}.$$
- By van der Corput there are $\gg \delta^{O(1)}N$ values of $h \in [N]$ such that
$$\left|\mathrm{E}_{n\in[N]}[F(a^{n+h}\Gamma)\overline{F(a^n\Gamma)}]\right| \gg \delta^{O(1)}.$$
- Given $g \in G$, write $g = \{g\}[g]$, where $\psi(\{g\}) \in [0,1)^3$ and $[g] \in \Gamma$. Hence the expectation is
$$\left|\mathrm{E}_{n\in[N]}[F(a^n\{a^h\}\Gamma)\overline{F(a^n\Gamma)}]\right| \gg \delta^{O(1)}.$$

□

Heisenberg case

Proof.

- Let $\tilde{F}_h : G^2/\Gamma^2 \to \mathbb{C}$ defined by

$$\tilde{F}_h(x,y) := F(\{a^h\}x)\overline{F(y)}.$$

Thus the expectation may be written

$$\left|\mathrm{E}_{n\in[N]}\left[\tilde{F}_h(\tilde{a}_h^n)\Gamma^2\right]\right| \gg \delta^{O(1)}$$

where $\tilde{a}_h := (\{a^h\}^{-1}a\{a^h\}, a)$.

- Notice $a^{-1}\{a^h\}^{-1}a\{a^h\} = \left[a, \{a^h\}\right] \in G_2$, so $\tilde{a}_h$ lies in the subgroup $G^\square = G \times_{G_2} G := \{(g,g') : g^{-1}g' \in G_2\}$ of G^2.

□

Proof.

- We can check that the commutator subgroup of $G^\square$ is $G_2^\Delta = \{(g_2, g_2) : g_2 \in G_2\}$ as follows. Let $(g, g'), (h, h') \in G^\square$. Then $[(g, g'), (h, h')] = ([g, h], [g', h']) \in G_2^2$.
- We have, since G_2 is in the center,

$$\begin{aligned}[g, h][g', h']^{-1} &= g^{-1}h^{-1}gh(h')^{-1}(g')^{-1}h'g' \\ &= h(h')^{-1}g(g')^{-1}h^{-1}h'g^{-1}g' = 1.\end{aligned}$$

□

Heisenberg case

Proof.

- A Mal'cev basis for $G^\square/\Gamma^\square$, $\mathscr{X}^\square = \{X_1^\square, X_2^\square, X_3^\square, X_4^\square\}$

$$X_1^\square = \begin{pmatrix} 0 & 1 & \{0,0\} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2^\square = \begin{pmatrix} 0 & 0 & \{0,0\} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, X_3^\square = \begin{pmatrix} 0 & 0 & \{1,0\} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_4^\square = \begin{pmatrix} 0 & 0 & \{1,1\} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where

$$\begin{pmatrix} 0 & x & \{y,y'\} \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} := \left(\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x & y' \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right).$$

- Projection onto the horizontal torus $\mathbb{R}^3/\mathbb{Z}^3$ is given by projection onto the first three coordinates.

□

Heisenberg case

Proof.

- Write $F_h^\square$ and $a_h^\square$ for the restrictions of $\tilde{F}_h$ and $\tilde{a}_h$ to $G^\square$ and write $\Gamma^\square := \Gamma \times_{\Gamma \cap G_2} \Gamma$. Integrating in the $X_3^\square$ direction shows that $F_h^\square$ is mean zero.
- Check that

$$\begin{aligned} F_h^\square((g_2, g_2) \cdot (g, g')) &= F(\{a^h\} g_2 g)\overline{F(g_2 g')} \\ &= \xi(g_2)\overline{\xi(g_2)} F(\{a^h\} g)\overline{F(g')} = F_h^\square((g, g')) \end{aligned}$$

  so $F_h^\square$ is $[G^\square, G^\square]$-invariant, and so factors through the projection $\pi^\square$ to the abelianization.
- Write $F_h' : \mathbb{R}^3/\mathbb{Z}^3 \to \mathbb{C}$ defined by $F_h'(\pi^\square(x)) = F_h^\square(x\Gamma^\square)$. We have F_h' is mean zero and has Lipschitz norm bounded by 1.

□

Heisenberg case

Proof.

- Since
$$\left|\mathbb{E}_{n\in[N]}[F'_h(n\pi^\square(a_h^\square))]\right| \gg \delta^{O(1)}$$
we obtain that for $\gg \delta^{O(1)}N$ values of h there exists $k_h^\square \in \mathbb{Z}^3$, $0 < |k_h^\square| \ll \delta^{-O(1)}$ such that
$$\|k_h^\square \cdot \pi^\square(a_h^\square)\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{\delta^{-O(1)}}{N}.$$
- Picking the most common values of $k_h^\square$, the same conclusion holds for a single $k^\square$. Let $\eta : G^\square/\Gamma^\square \to \mathbb{R}/\mathbb{Z}$ be defined as $\eta(x) := k^\square \cdot \pi^\square(x)$.

□

Heisenberg case

Proof.

- Notice that by decomposing along the first two coordinates in the Mal'cev basis, we can write $\eta(g', g) = \eta_1(g) + \eta_2(g' g^{-1})$ where η_1 is a horizontal character of G and $\eta_2 : G_2/(\Gamma \cap G_2) \to \mathbb{R}/\mathbb{Z}$.
- Calculate $\eta(\tilde{a}_h) = \eta_1(a) + \eta_2([a, \{a^h\}])$.
- In coordinates, if $\psi(x) = (t_1, t_2, t_3)$ and $\psi(y) = (u_1, u_2, u_3)$ then $\psi([x, y]) = (0, 0, t_1 u_2 - t_2 u_1)$. If $\psi(a) = (\gamma_1, \gamma_2, *)$ then $\psi(\{a^h\}) = (\{\gamma_1 h\}, \{\gamma_2 h\}, *)$.
- Set $\gamma := (\gamma_1, \gamma_2) = \pi(a)$, $\zeta := (-\gamma_2, \gamma_1)$ and observe $\eta(\tilde{a}_h) = k_1 \cdot \gamma + k_2 \zeta \cdot \{\gamma h\}$ so that for $\gg \delta^{O_m(1)} N$ values of $h \in [N]$

$$\|k_1 \cdot \gamma + k_2 \zeta \cdot \{\gamma h\}\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{\delta^{-O(1)}}{N}.$$

□

Bracket polynomial lemma

The proof is now completed by the following 'bracket polynomial lemma.'

Lemma

Let $\delta \in (0,1)$ and let $N \geqslant 1$ be an integer. Let $\theta \in \mathbb{R}$, $\gamma \in \mathbb{R}^2/\mathbb{Z}^2$ and $\zeta \in \mathbb{R}^2$ satisfy $|\zeta| \ll 1$. Suppose that for at least δN values of $h \in [N]$, we have

$$\|\theta + \zeta \cdot \{\gamma h\}\|_{\mathbb{R}/\mathbb{Z}} \leqslant \frac{1}{\delta N}.$$

Then either $\theta, |\zeta| \ll \frac{\delta^{-O(1)}}{N}$ or else there is $k \in \mathbb{Z}^2$, $0 < \|k\| \ll \delta^{-O(1)}$ such that $\|k \cdot \gamma\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{\delta^{-O(1)}}{N}$.

In either case we obtain a horizontal character k, $0 < \|k\| \ll \delta^{-O(1)}$ on G/Γ satisfying $\|k \cdot \gamma\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{\delta^{-O(1)}}{N}$.

Bracket polynomial lemma

Proof.

- We may assume that

$$\|\theta + \zeta \cdot \{\gamma h\}\|_{\mathbb{R}/\mathbb{Z}} \leqslant \frac{\delta}{10}\|\zeta\|_\infty$$

for at least δN values $h \in [N]$, or otherwise the first conclusion holds.

- Define

$$\Omega := \left\{t \in \mathbb{R}^2/\mathbb{Z}^2 : \|\theta + \zeta \cdot \{t\}\|_{\mathbb{R}/\mathbb{Z}} \leqslant \frac{\delta}{10}\|\zeta\|_\infty\right\}$$

and

$$\tilde{\Omega} := \left\{x \in \mathbb{R}^2/\mathbb{Z}^2 : d(x, \Omega) < \frac{\delta}{10}\right\}.$$

By slicing, one finds $|\tilde{\Omega}| < \frac{\delta}{2}$.

□

Bracket polynomial lemma

Proof.

- Define $F : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}_{\geqslant 0}$,

$$F(x) := \max\left(1 - \frac{10d(x,\Omega)}{\delta}, 0\right).$$

- Since F is 1 on Ω, $\mathbf{E}_{n\in[N]}[F(\gamma n)] \geqslant \delta$.
- Since F is supported on $\tilde{\Omega}$, $\int_{\mathbb{R}^2/\mathbb{Z}^2} F(x)dx < \frac{\delta}{2}$.
- Thus

$$\left|\mathbf{E}_{n\in[N]}[F(\gamma n)] - \int_{\mathbb{R}^2/\mathbb{Z}^2} F(x)dx\right| \geqslant \frac{\delta}{2}.$$

- Since $\|F\|_{\mathrm{Lip}} \ll \frac{1}{\delta}$ we find that $\{\gamma n\}_{n\in[N]}$ is not $c\delta^2$-equidistributed for some $c > 0$, and the conclusion follows.

□