Math 639: Lecture 11

Convergence of Martingales

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Submartingales

Theorem

If X_n is a submartingale and N is a stopping time with $\mathrm{Prob}(N \leqslant k) = 1$ then

$$\mathrm{E}[X_0] \leqslant \mathrm{E}[X_N] \leqslant \mathrm{E}[X_k].$$

Submartingales

Proof.

- $X_{N\wedge n}$ is a submartingale, so

$$E[X_0] = E[X_{N\wedge 0}] \leqslant E[X_{N\wedge k}] = E[X_N].$$

- Let $K_n = \mathbf{1}_{N<n}$. Since K_n is predictable, $(K\cdot X)_n = X_n - X_{N\wedge n}$ is a submartingale, so

$$E[X_k] - E[X_N] = E[(K\cdot X)_k] \geqslant E[(K\cdot X)_0] = 0.$$

□

Doob's inequality

Theorem (Doob's inequality)

Let X_m be a submartingale,

$$\overline{X}_n = \max_{0 \leqslant m \leqslant n} X_m^+,$$

$\lambda > 0$, *and* $A = \{\overline{X}_n \geqslant \lambda\}$. *Then*

$$\lambda \operatorname{Prob}(A) \leqslant \mathrm{E}[X_n \mathbf{1}_A] \leqslant \mathrm{E}[X_n^+].$$

Doob's inequality

Proof.

- Let $N = \inf\{m : X_m \geqslant \lambda \text{ or } m = n\}$. Since $X_N \geqslant \lambda$ on A,

$$\lambda \,\mathrm{Prob}(A) \leqslant \mathrm{E}[X_N \mathbf{1}_A] \leqslant \mathrm{E}[X_n \mathbf{1}_A].$$

$\square$

Example

- Let $S_n = \xi_1 + \cdots + \xi_n$ where the ξ_m are independent and have $\mathrm{E}[\xi_m] = 0$, $\sigma_m^2 = \mathrm{E}[\xi_m^2] < \infty$.
- We have $X_n = S_n^2$ is a submartingale.
- Choosing $\lambda = x^2$ in the previous theorem, we get Kolmogorov's maximal inequality

$$\mathrm{Prob}\left(\max_{1 \leqslant m \leqslant n} |S_m| \geqslant x\right) \leqslant x^{-2}\,\mathrm{Var}(S_n).$$

L^p maximum inequality

Theorem

If X_n is a submartingale, then for $1 < p < \infty$,

$$\mathrm{E}[\overline{X}_n^p] \leqslant \left(\frac{p}{p-1}\right)^p \mathrm{E}[X_n^+]^p.$$

Proof.

Calculate

$$\begin{aligned}
\mathrm{E}\left[|\overline{X}_n \wedge M|^p\right] &= \int_0^\infty p\lambda^{p-1}\,\mathrm{Prob}(\overline{X}_n \wedge M \geqslant \lambda)d\lambda \\
&\leqslant \int_0^\infty p\lambda^{p-1}\left(\lambda^{-1}\int X_n^+ \mathbf{1}\left(\overline{X}_n \wedge M \geqslant \lambda\right) dP\right) d\lambda \\
&= \int X_n^+ \int_0^{\overline{X}_n \wedge M} p\lambda^{p-2} d\lambda dP \\
&= \frac{p}{p-1}\int X_n^+\left(\overline{X}_n \wedge M\right)^{p-1} dP \\
&\leqslant \frac{p}{p-1}\mathrm{E}[|X_n^+|^p]^{\frac{1}{p}}\,\mathrm{E}[|\overline{X}_n \wedge M|^p]^{\frac{p-1}{p}}.
\end{aligned}$$

The result follows on letting $M \uparrow \infty$. □

L^1 maximum inequality

Theorem

Let X_n be a submartingale and $\log^+ x = \max(\log x, 0)$*.*

$$\mathrm{E}[\overline{X}_n] \leqslant (1 - e^{-1})^{-1}\left[1 + \mathrm{E}[X_n^+ \log^+(X_n^+)]\right].$$

Proof.

Exercise. $\square$

L^p convergence theorem

Theorem

If X_n is a martingale with $\sup \mathrm{E}[|X_n|^p] < \infty$ where $p > 1$, then $X_n \to X$ a.s. and in L^p.

L^p convergence theorem

Proof.

- $(\mathrm{E}[X_n^+])^p \leqslant (\mathrm{E}[|X_n|])^p \leqslant \mathrm{E}[|X_n|^p]$. Hence $X_n \to X$ a.s.
- By the L^p maximum inequality,

$$\mathrm{E}\left[\left(\sup_{0 \leqslant m \leqslant n} |X_m|\right)^p\right] \leqslant \left(\frac{p}{p-1}\right)^p \mathrm{E}[|X_n|^p].$$

  Letting $n \to \infty$, $\sup |X_n| \in L^p$, so $\mathrm{E}[|X_n - X|^p] \to 0$ by dominated convergence.

□

Orthogonality of martingale increments

Theorem

Let X_n be a martingale with $\mathrm{E}[X_n^2] < \infty$ for all n. If $m \leqslant n$ and $Y \in \mathscr{F}_m$ has $\mathrm{E}[Y^2] < \infty$, then

$$\mathrm{E}[(X_n - X_m)Y] = 0.$$

Orthogonality of martingale increments

Proof.

By Cauchy-Schwarz, $\mathrm{E}[|(X_n - X_m)Y|] < \infty$, so

$$\mathrm{E}[(X_n - X_m)Y] = \mathrm{E}[Y\,\mathrm{E}[(X_n - X_m)|\mathscr{F}_m]] = 0.$$

$\square$

Conditional variance formula

Theorem

If X_n is a martingale with $\mathrm{E}[X_n^2] < \infty$ for all n, then

$$\mathrm{E}[(X_n - X_m)^2 | \mathscr{F}_m] = \mathrm{E}[X_n^2 | \mathscr{F}_m] - X_m^2.$$

Conditional variance formula

Proof.

Calculate

$$\begin{aligned} \mathrm{E}[X_n^2 - 2X_nX_m + X_m^2 | \mathscr{F}_m] &= \mathrm{E}[X_n^2|\mathscr{F}_m] - 2X_m \mathrm{E}[X_n|\mathscr{F}_m] + X_m^2 \\ &= \mathrm{E}[X_n^2|\mathscr{F}_m] - X_m^2. \end{aligned}$$

□

Square integrable martingales

Definition

Let X_n be a martingale with $X_0 = 0$ and $\mathrm{E}[X_n^2] < \infty$ for all n. Thus X_n^2 is a sub-martingale. Write $X_n^2 = M_n + A_n$ where M_n is a martingale. A_n is called the *increasing process*. Let $A_\infty = \lim A_n$.

Square integrable martingales

Theorem

We have $\mathrm{E}[\sup_m |X_m|^2] \leqslant 4\,\mathrm{E}[A_\infty]$.

Proof.

The L^2 maximum inequality gives

$$\mathrm{E}\left[\sup_{0\leqslant m\leqslant n} |X_m|^2\right] \leqslant 4\,\mathrm{E}[X_n^2] = 4\,\mathrm{E}[A_n].$$

The conclusion follows from monotone convergence. $\square$

Square integrable martingales

Theorem

$\lim_{n\to\infty} X_n$ *exists and is finite a.s. on* $\{A_\infty < \infty\}$.

Proof.

- Let $a > 0$. Since $A_{n+1} \in \mathscr{F}_n$, $N = \inf\{n : A_{n+1} > a^2\}$.
- Since $A_{N\wedge n} \leqslant a^2$,
$$\mathrm{E}\left[\sup_n |X_{N\wedge n}|^2\right] \leqslant 4a^2.$$
- Hence $\lim X_{N\wedge n}$ exists and is finite a.s. Since this holds for all a, the result follows.

$\square$

Square integrable martingales

Theorem

Let $f \geqslant 1$ be increasing with $\int_0^\infty f(t)^{-2} dt < \infty$. Then $\frac{X_n}{f(A_n)} \to 0$ a.s. on $\{A_\infty = \infty\}$.

Square integrable martingales

Proof.

- Let $H_m = f(A_m)^{-1}$ is bounded and predictable so

$$Y_n = (H \cdot X)_n = \sum_{m=1}^{n} \frac{X_m - X_{m-1}}{f(A_m)}$$

  is a martingale.

- The increasing process associated to Y_n satisfies

$$\begin{aligned} B_{n+1} - B_n &= \mathrm{E}[(Y_{n+1} - Y_n)^2 | \mathscr{F}_n] \\ &= \mathrm{E}\left[\frac{(X_{n+1} - X_n)^2}{f(A_{n+1})^2} \middle| \mathscr{F}_n\right] = \frac{A_{n+1} - A_n}{f(A_{n+1})^2}. \end{aligned}$$

$\square$

Square integrable martingales

Proof.

- Since
$$\sum_{n=0}^{\infty} \frac{A_{n+1}-A_n}{f(A_{n+1})^2} \leqslant \sum_{n=0}^{\infty} \int_{[A_n, A_{n+1})} f(t)^{-2} dt < \infty.$$
  Hence $Y_n \to Y_\infty$ a.s.
- It follows that $\frac{X_n}{f(A_n)} \to 0$ a.s. by Kronecker's lemma.

$\square$

Uniformly integrable random variables

Definition

A collection of random variables $\{X_i : i \in I\}$ is *uniformly integrable* if

$$\lim_{M\to\infty}\left(\sup_{i\in I}\mathrm{E}[|X_i|\mathbf{1}(|X_i|>M)]\right)=0.$$

Choose M sufficiently large in the definition so that the sup is less than 1. Then

$$\sup_{i\in I}\mathrm{E}[|X_i|]\leqslant M+1<\infty.$$

Uniformly integrable random variables

Theorem

Given a probability space $(\Omega, \mathscr{F}_0, \mathrm{Prob})$ *and an* $X \in L^1$, *then* $\{\mathrm{E}[X|\mathscr{F}] : \mathscr{F} \subset \mathscr{F}_0\}$ *is uniformly integrable.*

Uniformly integrable random variables

Proof.

- If A_n is a sequence of sets with $\mathrm{Prob}(A_n) \to 0$, then $\mathrm{E}[|X|\mathbf{1}_{A_n}] \to 0$ by dominated convergence. Hence, for each $\epsilon > 0$ there exists $\delta > 0$ such that $\mathrm{Prob}(A_n) < \delta$ implies $\mathrm{E}[|X|\mathbf{1}_{A_n}] < \epsilon$.
- Apply Jensen's inequality to find, for $M > 0$,

$$\begin{aligned} \mathrm{E}\left[|\,\mathrm{E}[X|\mathscr{F}]|\mathbf{1}(|\,\mathrm{E}[X|\mathscr{F}]| > M)\right] &\leqslant \mathrm{E}[\mathrm{E}[|X||\mathscr{F}]\mathbf{1}(\mathrm{E}[|X||\mathscr{F}] > M)] \\ &= \mathrm{E}[|X|\mathbf{1}(\mathrm{E}[|X||\mathscr{F}] > M)]. \end{aligned}$$

- Choose M so that $\mathrm{E}[|X|] \leqslant M\delta$ so that

$$\mathrm{Prob}(\mathrm{E}[|X||\mathscr{F}] > M) \leqslant \frac{\mathrm{E}[\mathrm{E}[|X||\mathscr{F}]]}{M} = \frac{\mathrm{E}[|X|]}{M} \leqslant \delta.$$

- Thus $\mathrm{E}\left[|\,\mathrm{E}[X|\mathscr{F}]|\mathbf{1}(|\,\mathrm{E}[X|\mathscr{F}]| > M)\right] \leqslant \epsilon$ for all $\mathscr{F}$.

$\square$

Uniformly integrable random variables

Theorem

If $X_n \to X$ in probability, then the following are equivalent.

1. $\{X_n : n \geqslant 0\}$ *is uniformly integrable.*
2. $X_n \to X$ *in* L^1
3. $\mathrm{E}[|X_n|] \to \mathrm{E}[|X|] < \infty$.

Uniformly integrable random variables

Proof.

- 1 implies 2:
  - Let
$$\phi_M(x) = \begin{cases} M & x \geqslant M \\ x & |x| \leqslant M \\ -m & x \leqslant -M \end{cases}$$
    Thus
$$|X_n - X| \leqslant |X_n - \phi_M(X_n)| + |\phi_M(X_n) - \phi_M(X)| + |\phi_M(X) - X|.$$
  - Since $|\phi_M(Y) - Y| \leqslant |Y|\mathbf{1}(|Y| > M)$, taking expected values
$$\mathrm{E}[|X_n - X|] \leqslant \mathrm{E}[|\phi_M(X_n) - \phi_M(X)|] + \mathrm{E}[|X_n|\mathbf{1}(|X_n| > M)] + \mathrm{E}[|X|\mathbf{1}(|X| > M)].$$

□

Uniformly integrable random variables

Proof.

-
  - Since $\phi_M(X_n) \to \phi_M(X)$ in probability, the first term tends to 0.
  - The second term tends to 0 as M tends to ∞ by uniform integrability.
  - $\sup \mathrm{E}[|X_n|] < \infty$ implies $\mathrm{E}[|X|] < \infty$, which implies $\mathrm{E}[|X|\mathbf{1}(|X| > M)]$.
- 2 implies 3: Jensen gives

$$|\,\mathrm{E}[|X_n|] - \mathrm{E}[|X|]| \leqslant \mathrm{E}[||X_n| - |X||] \leqslant \mathrm{E}[|X_n - X|] \to 0.$$

$\square$

Uniformly integrable random variables

Proof.

- 3 implies 1:
  - Let ψ_M interpolate linearly between $f(x) = x$ on $[0, M-1]$ and 0 on $[M, \infty)$.
  - $\mathrm{E}[\psi_M(|X_n|)] \to \mathrm{E}[\psi_M(|X|)]$ by convergence in probability.
  - Choose M sufficiently large so that $\mathrm{E}[|X|] - \mathrm{E}[\psi_M(|X|)] \leqslant \frac{\epsilon}{2}$. If n is sufficiently large,

$$\mathrm{E}[|X_n|\mathbf{1}(|X_n| > M)] \leqslant \mathrm{E}[|X_n|] - \mathrm{E}[\psi_M(|X_n|)] < \epsilon.$$

$\square$

Uniformly integrable random variables

Theorem

For a submartingale, the following are equivalent.

1. *It is uniformly integrable.*
2. *It converges a.s. in L^1*
3. *It converges in L^1.*

Uniformly integrable random variables

Proof.

- 1 implies 2: Uniform integrability implies $\sup \mathrm{E}[|X_n|] < \infty$, so the martingale convergence theorem implies almost sure convergence. The convergence in L^1 follows from the previous theorem.
- 2 implies 3: This is automatic.
- 3 implies 1: Convergence in L^1 implies convergence in probability, so this follows from the previous theorem.

$\square$

Uniformly integrable random variables

Lemma

If integrable random variables $X_n \to X$ in L^1 then $\mathrm{E}[X_n \mathbf{1}_A] \to \mathrm{E}[X \mathbf{1}_A]$.

Proof.

$$|\,\mathrm{E}[X_m \mathbf{1}_A] - \mathrm{E}[X \mathbf{1}_A]| \leqslant \mathrm{E}[|X_m \mathbf{1}_A - X \mathbf{1}_A|] \leqslant \mathrm{E}[|X_m - X|] \to 0.$$

$\square$

Uniformly integrable random variables

Lemma

If a martingale $X_n \to X$ in L^1, then $X_n = \mathrm{E}[X|\mathscr{F}_n]$.

Proof.

- If $m > n$, $\mathrm{E}[X_m|\mathscr{F}_n] = X_n$, so if $A \in \mathscr{F}_n$, $\mathrm{E}[X_n \mathbf{1}_A] = \mathrm{E}[X_m \mathbf{1}_A]$
- Since $\mathrm{E}[X_m \mathbf{1}_A] \to \mathrm{E}[X \mathbf{1}_A]$ we have $\mathrm{E}[X_n \mathbf{1}_A] = \mathrm{E}[X \mathbf{1}_A]$ for all $A \in \mathscr{F}_n$. In particular, $X_n = \mathrm{E}[X|\mathscr{F}_n]$.

$\square$

Uniformly integrable random variables

Theorem

For a martingale, the following are equivalent.

1. *It is uniformly integrable*
2. *It converges a.s. and in* L^1
3. *It converges in* L^1
4. *There is an integrable random variable* X *so that* $X_n = \mathrm{E}[X|\mathscr{F}_n]$.

Proof.

The first two implications are as above. For 3 implies 4, this is the previous lemma. 4 implies 1 is a previous theorem. $\square$

Uniformly integrable random variables

Theorem

Suppose $\mathscr{F}_n \uparrow \mathscr{F}_\infty$ *and* $\mathscr{F}_\infty = \sigma\left(\bigcup_n \mathscr{F}_n\right)$. *As* $n \to \infty$, $\mathrm{E}[X|\mathscr{F}_n] \to \mathrm{E}[X|\mathscr{F}_\infty]$ *a.s. and in* L^1.

Uniformly integrable random variables

Proof.

- If $m > n$ then
$$\mathrm{E}[\mathrm{E}[X|\mathscr{F}_m]|\mathscr{F}_n] = \mathrm{E}[X|\mathscr{F}_n],$$
so $Y_n = \mathrm{E}[X|\mathscr{F}_n]$ is a martingale.
- Since Y_n is uniformly integrable, Y_n converges a.s. and in L^1 to a limit Y_∞.
- Observe $\mathrm{E}[X|\mathscr{F}_n] = Y_n = \mathrm{E}[Y_\infty|\mathscr{F}_n]$, and hence if $A \in \mathscr{F}_n$,
$$\int_A X dP = \int_A Y_\infty dP.$$
Since $\mathrm{E}[X|\mathscr{F}_\infty]$ and Y_∞ agree on a π-system in $\mathscr{F}_\infty$, they are equal there.

$\square$

Lévy's 0-1 law

Theorem

If $\mathscr{F}_n \uparrow \mathscr{F}_\infty$ *and* $A \in \mathscr{F}_\infty$ *then* $\mathrm{E}[\mathbf{1}_A|\mathscr{F}_n] \to \mathbf{1}_A$ *a.s.*

Dominated convergence

Theorem

Suppose $Y_n \to Y$ *a.s. and* $|Y_n| \leqslant Z$ *for all* n *where* $\mathrm{E}[Z] < \infty$. *If* $\mathscr{F}_n \uparrow \mathscr{F}_\infty$ *then*

$$\mathrm{E}[Y_n|\mathscr{F}_n] \to \mathrm{E}[Y|\mathscr{F}_\infty] \text{ a.s.}$$

Dominated convergence

Proof.

- Let $W_N = \sup\{|Y_n - Y_m| : n, m \geqslant N\}$. Note $\mathrm{E}[W_N] < \infty$.
- We have

$$\limsup_{n\to\infty} \mathrm{E}[|Y_n - Y||\mathscr{F}_n] \leqslant \lim_{n\to\infty} \mathrm{E}[W_n|\mathscr{F}_n] = \mathrm{E}[W_N|\mathscr{F}_\infty].$$

- Since $W_N \downarrow 0$ as $N \uparrow \infty$, $\mathrm{E}[W_N|\mathscr{F}_\infty] \downarrow 0$, and Jensen gives

$$|\mathrm{E}[Y_n|\mathscr{F}_n] - \mathrm{E}[Y|\mathscr{F}_n]| \leqslant \mathrm{E}[|Y_n - Y||\mathscr{F}_n] \to 0 \; a.s.$$

- Since $\mathrm{E}[Y|\mathscr{F}_n] \to \mathrm{E}[Y|\mathscr{F}_\infty]$ a.s. this suffices.

□

Backwards martingales

Definition

A *backwards martingale* is a martingale indexed by the negative integers, that is, X_n, $n \leqslant 0$, adapted to an increasing sequence of σ-algebras $\mathscr{F}_n$

$$\mathrm{E}[X_{n+1}|\mathscr{F}_n] = X_n, \qquad n \leqslant -1.$$

Backwards martingales

Theorem

$X_{-\infty} = \lim_{n\to-\infty} X_n$ exists a.s. and in L^1.

Proof.

- Let U_n be the number of upcrossings of $[a, b]$ by $X_{-n}, ..., X_0$.
- $(b - a)\,\mathrm{E}[U_n] \leqslant \mathrm{E}[(X_0 - a)^+]$.
- Letting $n \to \infty$, $\mathrm{E}[U_\infty] < \infty$, so the limit exists almost surely.
- Since $X_n = \mathrm{E}[X_0 | \mathscr{F}_n]$, X_n is uniformly integrable, so that the convergence is in L^1.

$\square$

Backwards martingales

Theorem

If $X_{-\infty} = \lim_{n\to-\infty} X_n$ *and* $\mathscr{F}_{-\infty} = \bigcap_n \mathscr{F}_n$*, then* $X_{-\infty} = \mathrm{E}[X_0|\mathscr{F}_{-\infty}]$.

Backwards martingales

Proof.

Since $X_n = \mathrm{E}[X_0|\mathscr{F}_n]$, if $A \in \mathscr{F}_{-\infty} \subset \mathscr{F}_n$,

$$\int_A X_n dP = \int_A X_0 dP.$$

Since $\mathrm{E}[X_n \mathbf{1}_A] \to \mathrm{E}[X_{-\infty}\mathbf{1}_A]$,

$$\int_A X_{-\infty} dP = \int_A X_0 dP$$

for all $A \in \mathscr{F}_{-\infty}$. $\square$

Backwards martingales

From the previous theorems it follows.

Theorem

If $\mathscr{F}_n \downarrow \mathscr{F}_{-\infty}$ as $n \downarrow -\infty$,

$$\mathrm{E}[Y|\mathscr{F}_n] \to \mathrm{E}[Y|\mathscr{F}_{-\infty}] \text{ a.s. and in } L^1.$$

Strong law of large numbers

Example

- Let $\xi_1, \xi_2, ...$ be i.i.d. with $\mathrm{E}[|\xi_i|] < \infty$.
- Let $S_n = \xi_1 + \cdots + \xi_n$, let $X_{-n} = \frac{S_n}{n}$, and let

$$\mathscr{F}_{-n} = \sigma(S_n, S_{n+1}, S_{n+2}, ...) = \sigma(S_n, \xi_{n+1}, \xi_{n+2}, ...).$$

- Calculate, using symmetry, for $j \leqslant n+1$,

$$\begin{aligned} \mathrm{E}[\xi_j|\mathscr{F}_{-n-1}] &= \frac{1}{n+1}\sum_{k=1}^{n+1} \mathrm{E}[\xi_k|\mathscr{F}_{-n-1}] \\ &= \frac{1}{n+1}\,\mathrm{E}[S_{n+1}|\mathscr{F}_n] = \frac{S_{n+1}}{n+1}. \end{aligned}$$

Strong law of large numbers

Example

- Since $X_{-n} = \frac{S_{n+1}-\xi_{n+1}}{n}$,

$$\begin{aligned} \mathrm{E}[X_{-n}|\mathscr{F}_{-n-1}] &= \mathrm{E}\left[\frac{S_{n+1}}{n}\middle|\mathscr{F}_{-n-1}\right] - \mathrm{E}\left[\frac{\xi_{n+1}}{n}\middle|\mathscr{F}_{-n-1}\right] \\ &= \frac{S_{n+1}}{n} - \frac{S_{n+1}}{n(n+1)} = \frac{S_{n+1}}{n+1} = X_{-n-1}. \end{aligned}$$

- Thus X_{-n} is a backwards martingale, and thus $\lim_{n\to\infty} \frac{S_n}{n} = \mathrm{E}[X_{-1}|\mathscr{F}_{-\infty}]$.
- Since $\mathscr{F}_{-n}$ has first n coordinates exchangeable, $\mathscr{F}_{-\infty} \subset \mathscr{E}$, and thus, by the Hewitt-Savage 0-1 law, $\lim_{n\to\infty} \frac{S_n}{n} = \mathrm{E}[X_{-1}]$ *a.s.*.

de Finetti's Theorem

A sequence $X_1, X_2, \ldots$ is said to be *exchangeable* if for each n and permutation π of $\{1, 2, \ldots, n\}$, $(X_1, \ldots, X_n)$ and $(X_{\pi(1)}, \ldots, X_{\pi(n)})$ have the same distribution.

Theorem

If $X_1, X_2, \ldots$ are exchangeable, then conditional on $\mathscr{E}$, $X_1, X_2, \ldots$ are independent and indentically distributed.

de Finetti's Theorem

Proof.

- Let ϕ be bounded, and introduce

$$A_n(\phi) = \frac{1}{(n)_k} \sum_i \phi(X_{i_1}, ..., X_{i_k})$$

where the sum runs over distinct sets $1 \leqslant i_1, ..., i_k \leqslant n$ and $(n)_k = n(n-1)\cdots(n-k+1)$.

- Calculate

$$\begin{aligned} A_n(\phi) = \mathrm{E}[A_n(\phi)|\mathscr{E}_n] &= \frac{1}{(n)_k} \sum_i \mathrm{E}[\phi(X_{i_1}, ..., X_{i_k})|\mathscr{E}_n] \\ &= \mathrm{E}[\phi(X_1, ..., X_k)|\mathscr{E}_n]. \end{aligned}$$

- It follows $A_n(\phi) \to \mathrm{E}[\phi(X_1, ..., X_k)|\mathscr{E}]$.

$\square$

de Finetti's Theorem

Proof.

- Let f and g be bounded on $\mathbb{R}^{k-1}$ and $\mathbb{R}$ and calculate

$$\begin{aligned}(n)_{k-1}A_n(f)nA_n(g) &= \sum_{\substack{1\leqslant i_1,\ldots,i_{k-1}\leqslant n\\ \text{distinct}}} f(X_{i_1},\ldots,X_{i_{k-1}})\sum_m g(X_m)\\ &= \sum_{\substack{1\leqslant i_1,\ldots,i_k\leqslant n\\ \text{distinct}}} f(X_{i_1},\ldots,X_{i_{k-1}})g(X_{i_k})\\ &+ \sum_{\substack{1\leqslant i_1,\ldots,i_{k-1}\leqslant n\\ \text{distinct}}}\sum_{j=1}^{k-1} f(X_{i_1},\ldots,X_{i_{k-1}})g(X_{i_j}).\end{aligned}$$

$\square$

de Finetti's Theorem

Proof.

- Let $\phi(x_1, \ldots, x_k) = f(x_1, \ldots, x_{k-1})g(x_k)$ and $\phi_j(x_1, \ldots, x_{k-1}) = f(x_1, \ldots, x_{k-1})g(x_j)$.
- Rearranging the above identity,

$$A_n(\phi) = \frac{n}{n-k+1} A_n(f)A_n(g) - \frac{1}{n-k+1}\sum_{j=1}^{k-1} A_n(\phi_j).$$

Letting $n \to \infty$,

$$\mathrm{E}[f(X_1, \ldots, X_{k-1})g(X_k)|\mathscr{E}] = \mathrm{E}[f(X_1, \ldots, X_{k-1})|\mathscr{E}]\,\mathrm{E}[g(X_k)|\mathscr{E}].$$

- The theorem now follows by induction.

$\square$

Theorem

If X_n is a uniformly integrable submartingale, then for any stopping time N, $X_{N \wedge n}$ is uniformly integrable.

Optional Stopping Theorems

Proof.

- X_n^+ is a submartingale, so $\mathrm{E}[X_{N\wedge n}^+] \leqslant \mathrm{E}[X_n^+]$.
- $\sup_n \mathrm{E}[X_{N\wedge n}^+] \leqslant \sup_n \mathrm{E}[X_n^+] < \infty$.
- By the martingale convergence theorem $X_{N\wedge n} \to X_N$ a.s. and $\mathrm{E}[|X_N|] < \infty$.
- Now calculate

$$\begin{aligned}\mathrm{E}[|X_{N\wedge n}|\mathbf{1}(|X_{N\wedge n}| > K)] &= \mathrm{E}[|X_N|\mathbf{1}(|X_N| > K, N \leqslant n)] \\ &\quad + \mathrm{E}[|X_n|\mathbf{1}(|X_n| > K, N > n)].\end{aligned}$$

Choosing K sufficiently large makes both parts on the right sufficiently small. $\square$

Optional Stopping Theorems

Theorem

If X_n is a uniformly integrable submartingale, then for any stopping time $N \leqslant \infty$ we have $E[X_0] \leqslant \mathrm{E}[X_N] \leqslant \mathrm{E}[X_\infty]$ where $X_\infty = \lim X_n$.

Optional Stopping Theorems

Proof.

Recall $\mathrm{E}[X_0] \leqslant \mathrm{E}[X_{N \wedge n}] \leqslant \mathrm{E}[X_n]$. Letting $n \to \infty$ and noting $X_{N \wedge n} \to X_N$ and $X_n \to X_\infty$ in L^1 proves the result. $\square$

Optional Stopping Theorems

Theorem (Optional stopping theorem)

If $L \leqslant M$ are stopping times and $Y_{M \wedge n}$ is a uniformly integrable submartingale, then $\mathrm{E}[Y_L] \leqslant \mathrm{E}[Y_M]$ and

$$Y_L \leqslant \mathrm{E}[Y_M | \mathscr{F}_L].$$

Optional Stopping Theorems

Proof.

- Set $X_n = Y_{M \wedge n}$ and use $\mathrm{E}[X_L] \leqslant \mathrm{E}[X_\infty]$ to obtain $\mathrm{E}[Y_L] \leqslant \mathrm{E}[Y_M]$.
- Let $A \in \mathscr{F}_L$ and

$$N = \begin{cases} L & \text{on } A \\ M & \text{on } A^c \end{cases}$$

We have $\mathrm{E}[Y_N] \leqslant \mathrm{E}[Y_M]$. Since $N = M$ on A^c,

$$\mathrm{E}[Y_L \mathbf{1}_A] \leqslant \mathrm{E}[Y_M \mathbf{1}_A] = \mathrm{E}[\mathrm{E}[Y_M | \mathscr{F}_L] \mathbf{1}_A].$$

Set $A_\epsilon = \{Y_L - \mathrm{E}[Y_M | \mathscr{F}_L] > \epsilon\}$ gives $\mathrm{Prob}(A_\epsilon) = 0$.

$\square$