PROBABILITY I, SPRING 2017, HW3

BOB HOUGH

Problem 1. Find independent random variables X, Y, Z so that Y and Z do not have the same distribution, but $X + Y$ and $X + Z$ have the same distribution.

Problem 2. Show that if $X_n = (X_n^1, ..., X_n^n)$ is uniformly distributed over the surface of the sphere of radius $\sqrt{n}$ in $\mathbb{R}^n$, then X_n^1 converges in distribution to a standard normal as $n \to \infty$. Hint: let $Y_1, Y_2, ...$ be i.i.d. standard normals, and let $X_n^i = Y_i \left(\frac{n}{\sum_{m=1}^n Y_m^2}\right)^{\frac{1}{2}}$.

Problem 3. Show that $\|\mu - \nu\|_{\mathrm{TV}} \leqslant 2\delta$ if and only if there are random variables X and Y with distributions μ and ν so that $\mathrm{Prob}(X \neq Y) \leqslant \delta$.

Problem 4. Let $X_1, X_2, ...$ be independent and set $S_n = X_1 + \cdots + X_n$. Show that if $|X_i| \leqslant M$ and $\sum_n \mathrm{Var}(X_n) = \infty$ then $(S_n - \mathbf{E}[S_n])/\sqrt{\mathrm{Var}(S_n)}$ converges in distribution to standard normal.

Problem 5. Let $X_1, X_2, ...$ be independent and set $S_n = X_1 + \cdots + X_n$. Suppose $\mathbf{E}[X_i] = 0$, $\mathbf{E}[X_i^2] = 1$, and $\mathbf{E}\left[|X_i|^{2+\delta}\right] \leqslant C$ for some $0 < \delta, C < \infty$. Then $S_n/\sqrt{n}$ converges in distribution to standard normal.