SPRING 2022: MAT 533 PRACTICE FINAL EXAM

Problem 1. *Let $a = (a_1, ..., a_n)$ be a vector whose coordinates are linearly independent over $\mathbb{Q}$. Prove that any Borel measurable set E of $(\mathbb{R}/\mathbb{Z})^n$ which is invariant under translation by a has measure 0 or 1.*

Problem 2. *Let $C : \mathscr{S}(\mathbb{R}^n) \to C(\mathbb{R}^n)$ be a continuous linear map which commutes with translation. Prove that C is given by convolution with a tempered distribution.*

Problem 3. *Form an $n \times n$ matrix by giving its entries independent mean 0 variance 1 standard Gaussian variables, then performing the Gram-Schmidt process on its columns treated as vectors. Prove that with probability 1 the resulting matrix is orthogonal, and that the distribution thus obtained is Haar measure on the orthogonal group.*

Problem 4. *A bi-infinite sequence $\{a_n\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is* positive definite *if, for all finite sets of complex numbers ϕ_n, $-N \leqslant n \leqslant N$, we have*

$$\sum_{n,k} a_{n-k}\phi_n\overline{\phi_k} \geqslant 0.$$

Prove that a sequence $\{a_n\}_{n\in\mathbb{Z}}$ is positive definite if and only if it is the set of Fourier coefficients of a positive measure on $\mathbb{R}/\mathbb{Z}$. (Hint: first show that these are the Fourier coefficients of a distribution.)

Problem 5. *Let $\Omega \subset \mathbb{C}$ be an open domain, and let $f : \Omega \to X$ be a map to a complex Banach space X. The function f is said to be* strongly analytic *if the difference quotients*

$$\lim_{k\to 0} \frac{1}{k}(f(x+k) - f(x))$$

exist at each point. The function f is said to be weakly analytic *if, for each bounded linear functional ℓ, $\ell(f(x))$ is an analytic function in the usual sense. Prove that f is strongly analytic if and only if it is weakly analytic. It may help to use the contour formula*

$$\begin{aligned}&\frac{1}{h-k}\left[\frac{\ell(f(z+h)) - \ell(f(z))}{h} - \frac{\ell(f(z+k)) - \ell(f(z))}{k}\right]\\ &= \frac{1}{2\pi i}\int_C \ell(f(\zeta))\frac{d\zeta}{(\zeta - z - h)(\zeta - z - k)(\zeta - z)}\end{aligned}$$

where C is a smooth contour with winding number 1 about $z, z+h, z+k$.