MATH 533, SPRING 2022, HW8

DUE APRIL 4

Problem 1. A distribution F is called harmonic if $\sum_1^n \partial_j^2 F = 0$. If F is harmonic and tempered, prove that F is a polynomial.

Problem 2. Let f be a continuous function on $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree $-n$, $f(rx) = r^{-n} f(x)$ for $r > 0$, and has mean 0 on the unit sphere. Show that f is not locally integrable near 0 unless $f = 0$, but f defines a tempered distribution $PV(f)$ by

$$\langle PV(f), \phi \rangle = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} f(x)\phi(x)dx.$$

The limit equals

$$\int_{|x|\leq 1} f(x)[\phi(x) - \phi(0)]dx + \int_{|x|>1} f(x)\phi(x)dx,$$

and these integrals converge absolutely.

Problem 3. On $\mathbb{R}$, let $F = PV((\pi x)^{-1})$. Check that

(1) $\hat{F}(\xi) = -i \operatorname{sgn} \xi$.
(2) The map $\phi \to F * \phi$, initially defined on Schwartz functions, extends to a unitary operator on L^2.

This is called the Hilbert transform.

Problem 4. Give $C^\infty(\mathbb{T}^n)$ the Fréchet space topology defined by the seminorms $\|\phi\|_{(\alpha)} = \|\partial^\alpha \phi\|_\infty$. The space $\mathcal{D}'(\mathbb{T}^n)$ of distributions on $\mathbb{T}^n$ is the space of continuous linear functionals on $C^\infty(\mathbb{T}^n)$, with the weak * topology. Prove the following.

(1) Distributions on $\mathbb{T}^n$ can be translated, differentiated, and multiplied by C^∞ functions, just as on $\mathbb{R}^n$.

(2) If $F \in \mathcal{D}'(\mathbb{T}^n)$, its Fourier transform is the function $\hat{F}$ on $\mathbb{Z}^n$ defined by $\hat{F}(\kappa) = \langle F, E_\kappa \rangle$ where $E_\kappa(x) = e^{-2\pi i \kappa \cdot x}$. Prove that a function g on $\mathbb{Z}^n$ is the Fourier transform of a distribution on $\mathbb{T}^n$ iff $|g(\kappa)| \leq C(1+|\kappa|)^N$ for some $C, N > 0$.

(3) If $F \in \mathcal{D}'(\mathbb{T}^n)$ and $\phi \in C^\infty(\mathbb{T}^n)$, prove $\langle F, \overline{\phi} \rangle = \sum_\kappa \hat{F}(\kappa)\overline{\hat{\phi}(\kappa)}$.

Problem 5. If $k \in \mathbb{N}$, H_k is the space of L^2 functions whose L^2 derivatives up to order k exist. Show that these strong derivatives coincide with the distribution derivatives.