MATH 533, SPRING 2022, HW4

DUE FEBRUARY 21

Prove the following statements regarding a Hilbert space $\mathcal{X}$.

Problem 1. (Polarization identity) For any $x, y \in \mathcal{X}$,

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2.$$

Problem 2. If $E \subset \mathcal{X}$, $(E^\perp)^\perp$ is the smallest closed subspace of $\mathcal{X}$ containing E.

Problem 3. Every closed convex set $K \subset \mathcal{X}$ has a unique element of minimal norm.

Problem 4. Let $\mathcal{X}$ be an infinite-dimensional Hilbert space.

(1) Every orthonormal sequence in $\mathcal{X}$ converges weakly to 0.
(2) The unit sphere $S = \{x : \|x\| = 1\}$ is weakly dense in the unit ball $B = \{x : \|x\| \leq 1\}$.

Problem 5. Let A and B be non-empty sets. Then $\ell^2(A)$ is isomorphic to $\ell^2(B)$ iff A and B have equal cardinality.

Problem 6. (The mean ergodic theorem) Let U be an unitary operator on the Hilbert space $\mathcal{X}$, $\mathcal{M} = \{x : Ux = x\}$, P the orthogonal projection onto $\mathcal{M}$, and $S_n = \frac{1}{n}\sum_0^{n-1} U^j$. Then $S_n \to P$ strongly.

In the following problems X is a locally compact Hausdorff space.

Problem 7. Let Y be a closed subset of X, and μ a Radon measure on Y. Prove that $I(f) = \int f|_Y d\mu$ is a positive linear functional on $C_c(X)$, and its induced Radon measure ν on X is given by $\nu(E) = \mu(E \cap Y)$.

Problem 8. Let μ be a Radon measure on X. Let N be the union of all open $U \subset X$ such that $\mu(U) = 0$. Then N is open, $\mu(N) = 0$, and if V is open and $V \setminus N \neq \emptyset$ then $\mu(V) > 0$. N^c is called the *support* of μ. Prove $x \in \operatorname{supp}(\mu)$ iff $\int f d\mu > 0$ for every $f \in C_c(X, [0,1])$ s.t $f(x) > 0$.