MATH 533, SPRING 2022, HW2

DUE FEBRUARY 7

Problem 1. Let $K \in C([0,1] \times [0,1])$. For $f \in C([0,1])$ define $Tf(x) = \int_0^1 K(x,y)f(y)dy$. Show that $Tf \in C([0,1])$, and $\{Tf : \|f\|_u \leq 1\}$ is precompact in $C([0,1])$.

Problem 2. Let U be an open subset of $\mathbb{C}$ and $\{f_n\}$ a sequence of holomorphic functions on U. If $\{f_n\}$ is uniformly bounded on compact subsets of U, show that there is a subsequence which converges uniformly to a holomorphic function on compact subsets of U.

Problem 3. Let X and Y be compact Hausdorff spaces. Show that the algebra generated by functions of the form $f(x,y) = g(x)h(y)$, where $g \in C(X)$ and $h \in C(Y)$, is dense in $C(X \times Y)$.

Problem 4. Let $(X, \mathcal{M})$ be a measurable space, and let $M(X)$ be the space of complex measures on $(X, \mathcal{M})$. Show that then $\|\mu\| = |\mu|(X)$ is a norm on $M(X)$ which makes $M(X)$ into a Banach space.

Problem 5. If $0 < \alpha \leq 1$, let $\Lambda_\alpha([0,1])$ be the set of Hölder continuous functions of exponent α on $[0,1]$. Thus $f \in \Lambda_\alpha([0,1])$ iff $\|f\|_{(\alpha)} < \infty$, where

$$\|f\|_{(\alpha)} = |f(0)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

(1) Show that $\|\cdot\|_{(\alpha)}$ is a norm which makes $\Lambda_\alpha([0,1])$ into a Banach space.
(2) Let $\lambda_\alpha([0,1])$ be the set of all $f \in \Lambda_\alpha([0,1])$ such that

$$\frac{|f(x) - f(y)|}{|x - y|^\alpha} \to 0, \qquad \text{as } x \to y, \text{ for all } y \in [0,1].$$

If $\alpha < 1$, show that $\lambda_\alpha([0,1])$ is a closed subspace of $\Lambda_\alpha([0,1])$, while if $\alpha = 1$, $\lambda_\alpha([0,1])$ contains only constant functions.

Problem 6. Show that a linear functional f on a normed vector space $\mathcal{X}$ is bounded iff $f^{-1}(\{0\})$ is closed.

Problem 7. If $\mathcal{M}$ is a finite-dimensional subspace of a normed vector space $\mathcal{X}$, show that there is a closed subspace $\mathcal{N}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = \mathcal{X}$.

Problem 8. If $\mathcal{X}$ is a Banach space, and $\mathcal{X}^*$ is separable, show that $\mathcal{X}$ is separable.