MATH 533, SPRING 2022, HW11

DUE MAY 2

Problem 1. Prove the following.

(1) If $f = \sum_1^n c_j \chi_{[a_j,b_j)}$ is a step function, define
$$I_f(\omega) = \int_0^\infty f(t) d\omega(t) = \sum_1^n c_j \left[\omega(b_j) - \omega(a_j)\right].$$
Then I_f is an L^2 random variable on Ω_c with mean 0 and variance $\|f\|_2^2 = \int |f|^2 dx$.

(2) The map $f \to I_f$ extends to an isometry from $L^2([0,\infty))$ to $L^2(\Omega_c)$.

(3) If $f \in BV([0,\infty))$ is right continuous and $\mathrm{supp}(f)$ is compact, there is a sequence $\{f_n\}$ of step functions such that $f_n \to f$ in L^2 and $df_n \to df$ vaguely.

(4) If $f \in BV([0,\infty))$ is right continuous and $\mathrm{supp}(f)$ is compact, then $I_f(\omega) = -\int_0^\infty \omega(t) df(t)$ almost surely.

Problem 2. Let R_x and L_x denote right and left translation by x in a locally compact group G. Let μ be a Radon measure on G, and $f \in C_c(G)$. Show that the functions $x \to \int L_x f d\mu$ and $x \to \int R_x f d\mu$ are continuous.

Problem 3. Let G be a locally compact group which is homeomorphic to an open subset U of $\mathbb{R}^n$ in such a way that, if we identify G with U, left translation is an affine map – that is, $xy = A_x(y) + b_x$ where A_x is a linear transformation of $\mathbb{R}^n$ and $b_x \in \mathbb{R}^n$. Show that $|\det A_x|^{-1} dx$ is a left Haar measure on G, where dx denotes Lebesgue measure on $\mathbb{R}^n$.

Problem 4. Let $G = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $x > 0$ and $y \in \mathbb{R}$. Construct a Borel set in G with finite left Haar measure but infinite right Haar measure. Construct a left uniformly continuous function on G that is not right uniformly continuous.

Problem 5. Let $\{G_\alpha\}_{\alpha\in A}$ be a family of topological groups and $G = \prod_{\alpha\in A} G_\alpha$. Prove that with product topology and coordinatewise multiplication, G is a topological group. If each G_α is compact and μ_α is the Haar measure on G_α such that $\mu_\alpha(G_\alpha) = 1$, then the Radon product of the $\mu_\alpha' s$ is a Haar measure on G.