

MATH 533, SPRING 2022, HW1

DUE JANUARY 31

Problem 1. Let $\mathcal{E} = \{(a, b] : -\infty < a < b < \infty\}$. Prove the following.

- (1) \mathcal{E} is a base for a topology \mathcal{T} on \mathbb{R} , in which the members of \mathcal{E} are both open and closed.
- (2) \mathcal{T} is first countable but not second countable.
- (3) \mathbb{Q} is dense in \mathbb{R} with respect to \mathcal{T} .

Problem 2. If X is a topological space, $x \in X$ is a *cluster point* of the sequence $\{x_j\}$ if for every neighborhood U of x , $x_j \in U$ for infinitely many j . Show that if X is first countable, x is a cluster point of $\{x_j\}$ iff some subsequence of $\{x_j\}$ converges to x .

Problem 3. Show that if X is a set, \mathcal{F} a collection of real-valued functions on X , and \mathcal{T} the weak topology generated by \mathcal{F} , then \mathcal{T} is Hausdorff iff for every $x, y \in X$ with $x \neq y$ there exists $f \in \mathcal{F}$ with $f(x) \neq f(y)$.

Problem 4. Show that if A is a countable set and X_α is a first (resp. second) countable space for each $\alpha \in A$, then $\prod_{\alpha \in A} X_\alpha$ is first (resp. second) countable.

Problem 5. Prove that a Hausdorff topological space X is normal iff X satisfies the conclusion of Urysohn's lemma.

Problem 6. Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in a topological space, and for each $\alpha \in A$ let $E_\alpha = \{x_\beta : \beta \succeq \alpha\}$. Show that x is a cluster point of $\langle x_\alpha \rangle$ iff $x \in \bigcap_{\alpha \in A} \overline{E_\alpha}$.

Problem 7. Prove that if X has the weak topology generated by a family \mathcal{F} of functions, then $\langle x_\alpha \rangle$ converges to $x \in X$ iff $\langle f(x_\alpha) \rangle$ converges to $f(x)$ for all $f \in \mathcal{F}$.

Problem 8. Let (X, \mathcal{T}) be a compact Hausdorff space. Prove that if \mathcal{T}' is another topology on X which is strictly stronger than \mathcal{T} , then \mathcal{T}' is Hausdorff

but not compact. If \mathcal{T}' is strictly weaker than \mathcal{T} , then \mathcal{T}' is compact but not Hausdorff.

Problem 9. If $x \in [0, 1]$, let $\sum_1^\infty a_n(x)2^{-n}$ [$a_n(x) = 0$ or 1] be the base-2 decimal expansion of x . Prove that the sequence $\langle a_n \rangle$ in $\{0, 1\}^{[0,1]}$ has no pointwise convergent subsequence.

Problem 10. Prove the following.

(1) If (Y, ρ) is a metric space then

$$\rho^*(y, z) = \frac{\rho(y, z)}{1 + \rho(y, z)}$$

is a bounded metric on Y which defines the same topology as ρ .

(2) If X is a topological space, the topology on \mathbb{C}^X of uniform convergence is induced by the metric

$$\rho(f, g) = \sup_{x \in X} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}.$$

(3) If X is a σ -compact LCH space and $\{U_n\}_1^\infty$ precompact open sets exhausting X , the topology on \mathbb{C}^X of uniform convergence on compact sets is induced by the metric

$$\rho(f, g) = \sum_1^\infty \sup_{x \in \bar{U}_n} \frac{2^{-n}|f(x) - g(x)|}{1 + |f(x) - g(x)|}.$$