MATH 533, SPRING 2020 MIDTERM

Each problem is worth 10 points.

Date: March 11, 2020.

Problem 1.

a. State and prove Bessel's inequality for a Hilbert space $\mathcal{H}$.
b. Using Bessel's inequality, or otherwise, prove that if $\mathcal{H}$ has a countable orthonormal basis, then any orthonormal basis of $\mathcal{H}$ is countable.

Solution.

a. Let $\{u_\alpha\}_{\alpha \in A}$ be an orthonormal set in $\mathcal{H}$. For any $x \in \mathcal{H}$,
$$\|x\|^2 \geq \sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2.$$
To prove this, calculate for any finite set $S \subset A$,
$$\begin{aligned} 0 &\leq \left\| x - \sum_{\alpha \in S} \langle x, u_\alpha \rangle u_\alpha \right\|^2 \\ &= \|x\|^2 - \sum_{\alpha \in S} |\langle x, u_\alpha \rangle|^2. \end{aligned}$$
In particular, for any x, the set of u_α with non-zero inner product with x is countable.

b. Let $\{u_n\}_{n \in \mathbb{N}}$ be a countable orthonormal basis of $\mathcal{H}$ and let $\{v_\alpha\}_{\alpha \in A}$ be another orthonormal basis. For each $n \in \mathbb{N}$, let $A_n = \{\alpha \in A : \langle v_\alpha, u_n \rangle \neq 0\}$, which is a countable set. By completeness of $\{u_n\}_{n \in \mathbb{N}}$, $A \subset \bigcup_n A_n$, which is countable.

Problem 2.

a. Let $\mathcal{X}$ be an infinite dimensional normed vector space. Prove that the unit ball $B_1 = \{x \in \mathcal{X} : \|x\| \leq 1\}$ is not compact in the norm topology.

b. Prove Alaoglu's Theorem: Let $\mathcal{X}$ be a Banach space. Prove that the unit ball in $\mathcal{X}^*$

$$B_1 = \{\ell \in \mathcal{X}^* : \|\ell\| \leq 1\}$$

is compact in the weak-* topology. (Hint: identify B_1 with a subset of $\prod_{x\in\mathcal{X}}[-\|x\|, \|x\|]$.)

Solution.

a. Form a sequence of unit vectors $x_1, x_2, ...$ as follows. Let x_1 be arbitrary. Having chosen $x_1, ..., x_n$, note that the span V_n of $x_1, ..., x_n$ is a closed subspace. Let $y \notin V_n$, and let $\delta = \inf_{x\in V_n} \|y - x\| > 0$. Choose $x \in V_n$ such that $\|y-x\| < 2\delta$ and set $x_{n+1} = \frac{y-x}{\|y-x\|}$. The sequence constructed satisfies, for $m > n$, the distance of x_m from V_n is at least $\frac{1}{2}$. It follows that no subsequence of $\{x_n\}$ is Cauchy, so B_1 is not compact.

b. (This applies to real Banach spaces, the modification for complex Banach spaces is straightforward.) Since $\|\ell\| \leq 1$, $|\ell(x)| \leq \|x\|$ and hence the map $\ell \mapsto \prod_{x\in\mathcal{X}} \ell(x)$ is an injection of B_1 into $\prod_{x\in\mathcal{X}}[-\|x\|, \|x\|]$, since $\mathcal{X}^*$ separates points. Furthermore, both the weak-* topology on $\mathcal{X}^*$ and the product topology correspond with the topology of pointwise convergence. Since $\prod_{x\in\mathcal{X}}[-\|x\|, \|x\|]$ is compact in the product topology by Tychonoff's theorem, it suffices to prove that the image of B_1 is a closed. Let $< \ell_\alpha >$ be a net in the image of B_1 converging to ℓ. For any $x, y \in \mathcal{X}$ and scalars a, b,

$$\ell(ax + by) = \lim \ell_\alpha(ax + by) = \lim a\ell_\alpha(x) + b\ell_\alpha(y) = a\ell(x) + b\ell(y).$$

Thus ℓ is linear and hence in the image of B_1.

Problem 3. Define the following sequence spaces of sequences of real numbers.

- For $p \geq 1$, $\ell_p = \{a = \{a_n\}_{n=1}^\infty : \|a\|_p^p = \sum_n |a_n|^p\}$
- $\ell_\infty = \{a = \{a_n\}_{n=1}^\infty : \|a\|_\infty = \sup_n |a_n|\}$
- $c_0 = \{a = \{a_n\} : \lim_n a_n = 0, \|a\|_\infty = \sup_n |a_n|\}$.

a. Prove that ℓ_p is separable, but ℓ_∞ is not.

b. Prove $c_0^* = \ell_1$, $\ell_1^* = \ell_\infty$ but $\ell_\infty^* \neq \ell_1$ by using Hahn-Banach. Give an example of a sequence in ℓ_1 which does not converge weakly, but converges weak-*.

Solution.

a. Let e_n be the nth standard basis vector. Let V be the rational linear span of $\{e_n\}_{n\in\mathbb{N}}$, that is, finite rational linear combinations of the e_j. This set is countable. To check that it is dense in ℓ_p, given $a \in \ell_p$ and $\epsilon > 0$, first approximate a with a' having finitely many non-zero entries, with $\|a - a'\|_p < \frac{\epsilon}{2}$. Then find a'' with finitely many rational entries such that $\|a' - a''\|_p < \frac{\epsilon}{2}$. Then $\|a - a''\|_p < \epsilon$. To check that ℓ_∞ is not separable, note that there are uncountably many 0-1 sequences, each of norm 1, and any two such sequences have ℓ_∞ distance 1. Any element of ℓ_∞ can have distance less than $\frac{1}{2}$ to at most one of these.

b. Given $a \in \ell_1$ and $b \in c_0$, let

$$(b, a) = \sum_n a_n b_n, \qquad |(b, a)| \leq \|b\|_\infty \|a\|_1.$$

Choosing a sequence $\{b^n\}$ from c_0 with $b_m^n = \operatorname{sgn}(a_m)$ if $m \leq n$, $b_m^n = 0$ if $m > n$ obtains a sequence of c_0 of norm 1 with $(b^n, a) \to \|a\|_1$. It follows that the norm of a as a linear functional is $\|a\|_1$, which embeds $\ell_1 \subset c_0^*$ isometrically. To prove that this is the whole space, given $\ell \in c_0^*$, let $a_n = (e_n, \ell)$, where e_n is the nth standard basis vector. By continuity, $(b, \ell) = \sum_n b_n a_n$. Choosing the sequence $\{b^n\}$ as before guarantees that $\sum |a_n| < \infty$ which identifies ℓ with an element of ℓ_1.

To check $\ell_1^* = \ell_\infty$, given $a \in \ell_\infty$, $b \in \ell_1$, define

$$(b, a) = \sum_n a_n b_n, \qquad |(b, a)| \leq \|a\|_\infty \|b\|_1.$$

The norm is achieved by selecting n such that $|a_n| = \|a\|_\infty$ and choosing $b = \pm e_n$. This embeds $\ell_\infty \subset \ell_1^*$ isometrically. Given any $\ell \in \ell_1^*$, define $a_n = (e_n, \ell)$. By continuity, $(b, \ell) = \sum_n a_n b_n$. Furthermore, $|a_n| \leq \|\ell\|$ so $a \in \ell_\infty$.

To check that $\ell_\infty^* \neq \ell_1$, note that ℓ_1 is separable, whereas ℓ_∞ is not. When the dual space of a Banach space $\mathcal{X}$ is separable, so is $\mathcal{X}$. To prove this using Hahn-Banach instead, suppose for contradiction that $\ell_\infty^* = \ell_1$. By identifying ℓ_∞ with ℓ_1^* we may assume that the pairing between ℓ_1 and ℓ_∞ is the usual one, which now identifies ℓ_1 with c_0^*. Extend linear functionals on c_0 to those on sequences with a finite limit, and from there to all of ℓ_∞, by Hahn-Banach. This is a contradiction, since ℓ_1 is determined by its pairing with c_0.

The sequence $\{e_n\} \subset \ell_1$ of standard basis vectors converges weak-* to 0, since each element of c_0 has limit 0. However, it does not converge weakly by pairing with $b_n = (-1)^n$ from ℓ_∞.

Problem 4. Let $\phi \in C_c^\infty(\mathbb{R}^n)$, $\int \phi = 1$, and for real $t > 0$, let $\phi_t(x) = t^{-n}\phi\left(\frac{x}{t}\right)$. Let $1 \le p < \infty$ and let $f \in L^p(\mathbb{R}^n)$. Prove that $\phi_t * f \in C^\infty(\mathbb{R}^n)$ and $\phi_t * f \to f$ in L^p as $t \downarrow 0$.

Solution. Use $\partial^\alpha f * \phi_t = f * \partial^\alpha \phi_t$, which is justified by dominated convergence, passing the derivatives under the integral sign. This verifies that $f * \phi_t$ is C^∞.

To check the convergence in L^p, write

$$f * \phi_t(x) - f(x) = \int [f(x-y) - f(x)]\phi_t(y)dt.$$

By Minkowski's inequality,

$$\|f * \phi_t - f\|_p \le \int |\phi_t(y)| \|f^y - f\|_p dy = \int |\phi(y)| \|f^{ty} - f\|_p dy.$$

This suffices, since ϕ has compact support and $f^{ty} \to f$ in L^p as $t \to 0$.

Problem 5. Let μ be a Radon measure on X. Prove that μ is inner regular on Borel sets of finite measure.

Solution. Let E be Borel measurable with $\mu(E) < \infty$. Given $\epsilon > 0$, since μ is outer regular at E, choose U open, $E \subset U$ with $\mu(U) < \mu(E) + \epsilon$. Since U is inner regular, choose K compact, $K \subset U$, with $\mu(K) > \mu(U) - \epsilon$. We have $\mu(U \setminus E) < \epsilon$, and hence we can find open $V \supset U \setminus E$ with $\mu(V) < \epsilon$. Let $F = K \setminus V$, which is compact and satisfies $F \subset E$. Then $\mu(F) + \mu(V) \geq \mu(K) \geq \mu(U) - \epsilon \geq \mu(E) - \epsilon$, so $\mu(F) \geq \mu(E) - 2\epsilon$. It follows that $\sup\{\mu(F) : F \subset E, \text{compact}\} = \mu(E)$.

Problem 6.

a. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $L(\mathcal{X}, \mathcal{Y})$ be the bounded linear maps between $\mathcal{X}$ and $\mathcal{Y}$. Give a neighborhood base at 0 for the strong and weak operator topologies.

b. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $T_n \in L(\mathcal{X}, \mathcal{Y})$ be such that, for each $x \in \mathcal{X}$, $\{T_n x\}$ is Cauchy. Prove that T_n converges strongly to some $T \in L(\mathcal{X}, \mathcal{Y})$.

Solution.

a. A base for the strong operator topology at 0 is, for $x_1, ..., x_n \in X$ and $\epsilon > 0$,

$$V_{x_1,...,x_n,\epsilon} = \{T : \|Tx_i\|_Y < \epsilon\}.$$

A base for the weak operator topology at 0 is, for $x_1, ..., x_n \in X, \ell_1, ..., \ell_m \in Y^*, \epsilon > 0$,

$$U_{x_1,...,x_n,\ell_1,...,\ell_m,\epsilon} = \{T : |(Tx_i, \ell_j)| < \epsilon\}.$$

b. Let, for each x, $Tx = \lim_n T_n x$. This definition is evidently linear. To check that it is bounded, note that $\|Tx\| = \lim_n \|T_n x\|$, so $\|T_n x\|$ is bounded pointwise, and hence $\|T_n\|$ is bounded by the uniform boundedness principle. Thus, for any y,

$$\|Ty\| = \lim_n \|T_n y\| \leq (\sup_n \|T_n\|)\|y\|.$$

Hence $T \in L(X, Y)$. Since $T_n x \to Tx$ for each x, $T_n \to T$ strongly.