MATH 533, SPRING 2020, HW9

DUE IN CLASS, APRIL 20

Problem 1. Let $G_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$, and if $f \in \mathcal{S}'(\mathbb{R}^n)$, let $u(x,t) = f * G_t(x)$. Prove

(1) u satisfies $(\partial_t - \Delta)u = 0$ on $\mathbb{R}^n \times (0,\infty)$ and $u(\cdot,t) \to f$ in $\mathcal{S}'$ as $t \to 0$.
(2) If f is a tempered function, then $u(x,t) \to f(x)$ a.e. as $t \to 0$.

Problem 2. Let W_t be the inverse Fourier transform of $(2\pi|\xi|)^{-1}\sin(2\pi|\xi|t)$. Prove the following.

(1) If $n = 1$, $W_t = \frac{1}{2}\chi(-t,t)$
(2) If $n = 3$, let σ_R denote surface measure on the sphere $|x| = R$. Then $\hat{\sigma}_R(\xi) = 2R|\xi|^{-1}\sin(2\pi R|\xi|)$, and hence $W_t = (4\pi t)^{-1}\sigma_t$.
(3) If $n = 2$, think of $\xi \in \mathbb{R}^2$ as an element of $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. Transform the integral

$$\frac{2R\sin(2\pi R|\xi|)}{|\xi|} = \int_{|x|=R} e^{-2\pi i x\cdot\xi} d\sigma_R(x)$$

as an integral over the disc $D_R = \{y : |y| \le R\}$ in $\mathbb{R}^2$. Conclude that for $n = 2$,

$$W_t(x) = (2\pi)^{-1}(t^2 - |x|^2)^{-\frac{1}{2}}\chi_{D_t}(x).$$

Problem 3. Solve $(\partial_t - \partial_x^2)u = 0$ on $(a,b) \times (0,\infty)$ with boundary conditions $u(x,0) = f(x)$ on (a,b), $u(a,t) = u(b,t) = 0$ for $t > 0$. Solve this again, but with boundary condition $u(a,t) = u(b,t) = 0$ replaced by $\partial_x u(a,t) = \partial_x u(b,t) = 0$.

Problem 4. If μ is a positive Borel measure on $\mathbb{T}^1$ with $\mu(\mathbb{T}^1) = 1$, show that $|\hat{\mu}(k)| < 1$ for all $k \neq 0$ unless μ is a convex combination of the point masses at $0, m^{-1}, ..., (m-1)m^{-1}$ for some $m \in \mathbb{N}$, in which case $\hat{\mu}(km) = 1$ for all $k \in \mathbb{Z}$.

Problem 5. Show that if $\Delta(\mathbb{R}^n)$ is the set of finite linear combinations of point masses on $\mathbb{R}^n$, then $\Delta(\mathbb{R}^n)$ is vaguely dense in $M(\mathbb{R}^n)$.