MATH 533, SPRING 2020, HW11

DUE IN CLASS, MAY 4

Problem 1. If $X_n \to X$ in probability, then $P_{X_n} \to P_X$ vaguely.

Problem 2. Identify $\mathbb{T}^1$ with $\{z \in \mathbb{C} : |z| = 1\}$.

(1) If $X_1, ..., X_n$ are independent, then $P_{X_1 X_2 \cdots X_n} = P_{X_1} * \cdots * P_{X_n}$.
(2) If $\{X_j\}$ is a sequence of independent random variables with common distribution λ, the distribution of $\prod_1^n X_j$ converges vaguely to the uniform distribution on $\mathbb{T}^1$ unless X_1 is supported on a finite subgroup of $\mathbb{T}^1$.

Problem 3. Given $b \in \mathbb{N} \setminus \{1\}$, let $B = \{0, 1, ..., b-1\}$ and $\Omega = B^{\mathbb{N}}$. Put the discrete topology on B and the product topology on Ω, and let P be the product measure on Ω, where each P_n is b^{-1} times counting measure on B. Let $\{X_n\}_1^\infty$ be the coordinate functions on Ω. Then if $A_1, ..., A_n \subset B$,

$$\mathbf{Prob}\left(\bigcap_1^n X_j^{-1}(A_j)\right) = b^{-n} \prod_1^n |A_j|$$

and $P(\{\omega\}) = 0$ for all $\omega \in \Omega$.

Problem 4. Prove the following. Let

$$\Omega' = \{\omega \in \Omega : X_n(\omega) \neq 0 \text{ for infinitely many } n\}.$$

Then $\Omega \setminus \Omega'$ is countable and $P(\Omega') = 1$. Define $F : \Omega \to [0, 1]$ by $F(\omega) = \sum_1^\infty X_n(\omega) b^{-n}$. Then $F|_{\Omega'}$ is a bijection from Ω' to $(0, 1]$ which maps $\mathcal{B}_{\Omega'}$ bijectively onto $\mathcal{B}_{(0,1]}$.

Problem 5. (Borel's normal number theorem) A number $x \in (0, 1]$ is called *normal* in base b if the digits $0, 1, ..., b-1$ occur with equal frequency in its base b decimal expansion, that is, if $n^{-1} X_j^{-1}(F^{-1}(x)) \to b^{-1}$ as $n \to \infty$. Almost every $x \in (0, 1]$ (with respect to Lebesgue measure) is normal in base b for every b.