MATH 533, SPRING 2020, HW10

DUE IN CLASS, APRIL 27

Problem 1. If $0 < p < 1$, let $\beta_p = p\delta_1 + (1-p)\delta_0$, and let $\beta_p^{(n)}$ be the nth convolution power of β_p,

$$\beta_p^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \delta_k.$$

If $a > 0$, let

$$\lambda_a = e^{-a} \sum_{0}^{\infty} \frac{a^k}{k!} \delta_k.$$

λ_a is called the *Poisson distribution* with parameter a. Prove the following.

(1) The mean and variance of λ_a are both a.
(2) $\lambda_a * \lambda_b = \lambda_{a+b}$.
(3) $\beta_{a/n}^{(n)}$ converges vaguely to λ_a as $n \to \infty$.

Problem 2. If $\sum_1^\infty n^{-2}\sigma_n^2 < \infty$, then $\lim n^{-2} \sum_1^n \sigma_j^2 = 0$. If $\{a_n\} \subset \mathbb{C}$ and $\lim a_n = a$, then $\lim n^{-1} \sum_1^n a_j = a$.

Problem 3. If $\{X_n\}$ is a sequence of independent random variables such that $\mathbf{E}(X_n) = 0$ and $\sum_1^\infty \sigma^2(X_n) < \infty$, then $\sum_1^\infty X_n$ converges almost surely.

Problem 4. If $\{X_n\}$ is a sequence of i.i.d. random variables which are not in L^1, then $\limsup n^{-1} |\sum_1^n X_j| = \infty$ almost surely.

Problem 5. (Shannon's Theorem) Let $\{X_i\}$ be a sequence of independent random variables on the sample space Ω having the common distribution $\lambda = \sum_1^r p_j \delta_j$ where $0 < p_j < 1$, $\sum_1^r p_j = 1$, and δ_j is the point mass at j. Define random variables $Y_1, Y_2, ...$ on Ω by

$$Y_n(\omega) = P\left(\{\omega' : X_j(\omega') = X_j(\omega) \text{ for } 1 \le j \le n\}\right).$$

Prove the following.

(1) $Y_n = \prod_1^n p_{X_j}$.
(2) $n^{-1} \log Y_n \to \sum_1^r p_j \log p_j$ almost surely.