SPRING 2020: MAT 533 FINAL EXAM

Solve the problems alone. You may consult the course textbook, but not other outside sources. Submit your solutions on Blackboard as for the weekly homework. Each problem is worth 25 points.

Problem 1. *Let $a = (a_1, ..., a_n)$ be a vector whose coordinates are linearly independent over $\mathbb{Q}$. Prove that any Borel measurable set E of $(\mathbb{R}/\mathbb{Z})^n$ which is invariant under translation by a has measure 0 or 1.*

Problem 2. *Let $C : \mathscr{S}(\mathbb{R}^n) \to C(\mathbb{R}^n)$ be a continuous linear map which commutes with translation. Prove that C is given by convolution with a tempered distribution.*

Problem 3. *Let $e_1, ..., e_d$ be the standard basis vectors of $\mathbb{R}^d$. Let μ be the probability measure on $\mathbb{Z}^d$ which assigns equal probability $\frac{1}{2d}$ to $\pm e_i$. The convolution $\mu^{*n} = \mu * \mu^{*(n-1)}$ has the distribution of n steps of simple random walk on $\mathbb{Z}^d$. Let X_n be the location of the walker at step n. Prove that as $n \to \infty$,*

$$p_{2n} = \mu^{*2n}(0) = (2 + o(1)) \left(\sqrt{\frac{d}{4\pi n}} \right)^d .$$

(Hint: express the return probability as an integral

$$p_{2n} = \int_{(\mathbb{R}/\mathbb{Z})^d} \hat{\mu}(\xi)^{2n} d\xi$$

and consider separately frequencies ξ of size roughly $\frac{1}{\sqrt{n}}$, between $\frac{1}{\sqrt{n}}$ and $O(1)$ and $O(1)$.)
Conclude that in dimensions $d \geqslant 3$ simple random walk returns to 0 finitely often with probability 1, and hence is transcient.
Let τ_k be the kth return time to 0 of simple random walk. Let

$$V = \sum_{m=1}^{\infty} \mathbf{1}(X_m = 0) = \sum_{k=1}^{\infty} \mathbf{1}(\tau_k < \infty)$$

be the number of visits to 0 of simple random walk. Calculate $\mathbf{E}[V]$ in two ways to prove that simple random walk in 1 or 2 dimensions returns to 0 infinitely often with probability 1, and hence is recurrent.

Problem 4. *Prove the mean ergodic theorem: Let U be a unitary map of a Hilbert space $\mathscr{H}$ and let P be projection onto the fixed space $\{\psi : U\psi = \psi\}$. Prove that $\frac{1}{N}(I + U + U^2 + \cdots + U^{N-1})f$ converges to Pf as $N \to \infty$.*

Problem 5. *Form an $n \times n$ matrix by giving its entries independent mean 0 variance 1 standard Gaussian variables, then performing the Gram-Schmidt process on its columns treated as vectors. Prove that with probability 1 the resulting matrix is orthogonal, and that the distribution thus obtained is Haar measure on the orthogonal group.*

Problem 6. *A bi-infinite sequence $\{a_n\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is* positive definite *if, for all finite sets of complex numbers ϕ_n, $-N \leqslant n \leqslant N$, we have*

$$\sum_{n,k} a_{n-k}\phi_n\overline{\phi_k} \geqslant 0.$$

Prove that a sequence $\{a_n\}_{n\in\mathbb{Z}}$ is positive definite if and only if it is the set of Fourier coefficients of a positive measure on $\mathbb{R}/\mathbb{Z}$. (Hint: first show that these are the Fourier coefficients of a distribution.)

Problem 7. *Let $\Omega \subset \mathbb{C}$ be an open domain, and let $f : \Omega \to X$ be a map to a complex Banach space X. The function f is said to be* strongly analytic *if the difference quotients*

$$\lim_{k\to 0}\frac{1}{k}(f(x+k) - f(x))$$

exist at each point. The function f is said to be weakly analytic *if, for each bounded linear functional ℓ, $\ell(f(x))$ is an analytic function in the usual sense. Prove that f is strongly analytic if and only if it is weakly analytic. It may help to use the contour formula*

$$\frac{1}{h-k}\left[\frac{\ell(f(z+h)) - \ell(f(z))}{h} - \frac{\ell(f(z+k)) - \ell(f(z))}{k}\right]$$
$$= \frac{1}{2\pi i}\int_C \ell(f(\zeta))\frac{d\zeta}{(\zeta - z - h)(\zeta - z - k)(\zeta - z)}$$

where C is a smooth contour with winding number 1 about $z, z+h, z+k$.