MATH 342, SPRING 2025 MIDTERM

Each problem is worth 10 points.

Problem 1.

a. Express in polar form $(1-i)^{200}$.
b. Find all of the roots of $z^8 = -1$
c. Express in rectangular coordinates $\frac{3+4i}{2+i}$.

Solution.

a. We have $(1-i) = \sqrt{2}e^{-\frac{i\pi}{4}}$ so $(1-i)^{200} = 2^{100}$.
b. Since $(-1) = \exp((2n+1)\pi i)$, $z = \exp((2n+1)\pi i/8)$ where $n \in \mathbb{Z}$. The choices $n = 0, 1, 2, 3, 4, 5, 6, 7$ cover all solutions.
c. $\frac{3+4i}{2+i} = \frac{(3+4i)(2-i)}{5} = \frac{10+5i}{5} = 2 + i$.

Problem 2.

a. Solve $\tan z = 2i$.
b. Find all values of i^π.
c. Find the value of $\cos(1 + 2\pi i)$.

Solution.

a. Write this as $\frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})} = 2i$ or $e^{iz} - e^{-iz} = -2(e^{iz} + e^{-iz})$ or $3e^{2iz} = -1$ or $e^{2iz} = -\frac{1}{3}$. Thus $z = \frac{1}{2i} \log -\frac{1}{3}$ or $z = \frac{1}{2i} \left((2n + 1)\pi i - \ln 3\right)$, $n \in \mathbb{Z}$ or

$$z = \frac{i}{2} \ln 3 + \left(n + \frac{1}{2}\right) \pi, n \in \mathbb{Z}.$$

Notice the solution is π-periodic since this is true of tangent.

b. $i = \exp\left(\left(\frac{1}{2} + 2n\right) \pi i\right)$, $n \in \mathbb{Z}$, so $i^\pi = \exp\left(\left(\frac{1}{2} + 2n\right) \pi^2 i\right)$, $n \in \mathbb{Z}$.

c. $\cos(1+2\pi i) = \frac{e^{(1+2\pi i)i}+e^{-(1+2\pi i)i}}{2} = \frac{e^{-2\pi+i}+e^{2\pi-i}}{2} = \frac{e^{-2\pi}(\cos 1+i \sin 1)+e^{2\pi}(\cos 1-i \sin 1)}{2} = \cos 1 \cosh(2\pi) - i \sin 1 \sinh(2\pi)$.

Problem 3.

a. Is the following function analytic? $f(x+iy) = x^3 - 3xy^2 + x^2 - y^2 + i(3x^2y - y^3 + 2xy)$

b. Recall a real valued function $H(x,y)$ is harmonic if H_{xx}, H_{yy} exist and are continuous and $H_{xx} + H_{yy} = 0$. Is $\log(x^2+y^2)$ harmonic in $(x,y) \neq (0,0)$?

Solution.

a. The function is $z^3 + z^2$, which is a polynomial, so analytic.

b. The function is $\Re \log(z^2)$, and being the real part of an analytic function is harmonic.

Problem 4. Let $f(x+iy) = x^2+y+ix$ and let C be the straight line contour starting from 0 to 1, followed by from 1 to $1+i$. Calculate $\int_C f(z)dz$.

Solution. Let C_1 be the segment from 0 to 1 parameterized by $z(t) = t, 0 \leq t \leq 1, z'(t) = 1,\ f(z(t)) = t^2 + it$. Let C_2 be the segment from 1 to $1+i$ parameterized by $z(t) = 1 + it$, $0 \leq t \leq 1$, $z'(t) = i$, $f(z(t)) = (1+i) + t$. Thus

$$\int_{C_1} f(z)dz = \int_0^1 t^2{+}itdt = \frac{1}{3}{+}\frac{i}{2}, \ \int_{C_2} f(z)dz = \int_0^1 (1{+}i{+}t)(i)dt = -1{+}i{+}\frac{i}{2}.$$

The combined integral is $-\frac{2}{3} + 2i$.

Problem 5. Let $C = \{2e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ and let $f(z) = \frac{e^z}{(z-1)^2}$. Calculate $\int_C f(z)dz$.

Solution. Let $g(z) = e^z$. By the derivative form of Cauchy's integral formula the integral is $2\pi i g'(1) = 2\pi i e$.