MATH 322, SPRING 2019 FINAL EXAM

MAY 14

Solve problems 1 and 2, and four of problems 3-8. Each problem is worth 25 points.

Problem 1. Let $\omega = \ln(x^2+y^2+z^2)dx \wedge dy + (x^2+y^2+z^2)dx \wedge dz + e^{x^2+y^2+z^2}dy \wedge dz$. Calculate $d\omega$.

Solution. We have

$$\begin{aligned} d\omega &= \frac{\partial}{\partial z}(\ln(x^2+y^2+z^2))dz \wedge dx \wedge dy + \frac{\partial}{\partial y}(x^2+y^2+z^2)dy \wedge dx \wedge dz \\ &\quad + \frac{\partial}{\partial x}(e^{x^2+y^2+z^2})dx \wedge dy \wedge dz \\ &= \left(\frac{2z}{x^2+y^2+z^2} - 2y + 2xe^{x^2+y^2+z^2}\right) dx \wedge dy \wedge dz. \end{aligned}$$

Problem 2. Let $\alpha : \mathbb{R} \to \mathbb{R}^n$ be given by $\alpha(t) = \begin{pmatrix} t \\ t^2 \\ \vdots \\ t^n \end{pmatrix}$. Let $\omega = \sum_{i=1}^n x_i dx_i$. Calculate $\alpha^*\omega$.

Solution. We have

$$\begin{aligned}
\alpha^*\omega &= \alpha^* \left(\sum_{i=1}^n x_i dx_i \right) \\
&= \sum_{i=1}^n \alpha_i(t) d\alpha_i(t) \\
&= \sum_{i=1}^n t^i (it^{i-1}) dt \\
&= \sum_{i=1}^n it^{2i-1} dt.
\end{aligned}$$

Problem 3. Let M be a compact oriented n manifold with boundary in $\mathbb{R}^n$ given the usual orientation. Let $\Theta_n = \frac{1}{n}\sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots dx_n$, where the hat indicates that index i term is omitted. Prove that

$$\int_{\partial M} \Theta_n = \mathrm{Vol}(M).$$

Let $\theta_n = \frac{1}{\|x\|^n}\Theta_n$ on $\mathbb{R}^n \setminus \{0\}$. Prove that θ_n is closed but not exact.

Solution. We have

$$\begin{aligned} d\Theta_n &= \frac{1}{n}\sum_{i=1}^n (-1)^{i-1} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots dx_n \\ &= dx_1 \wedge \cdots \wedge dx_n. \end{aligned}$$

Hence the first claim follows by Stokes' Theorem.

Write $\theta_n = \frac{1}{\|x\|^n} \wedge \Theta_n$ so that $d\theta_n = d\left(\frac{1}{\|x\|^n}\right) \wedge \Theta_n + \frac{1}{\|x\|^n} d\Theta_n$. Since $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$,

$$d\left(\frac{1}{(x_1^2 + \cdots + x_n^2)^{\frac{n}{2}}}\right) = -\frac{n}{\|x\|^{n+2}} \sum_{i=1}^n x_i dx_i.$$

It follows that

$$\begin{aligned} &d\left(\frac{1}{(x_1^2 + \cdots + x_n^2)^{\frac{n}{2}}}\right) \wedge \Theta_n \\ &= -\frac{1}{\|x\|^{n+2}} \sum_{i=1}^n (-1)^{i-1} x_i^2 dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots dx_n \\ &= -\frac{1}{\|x\|^n} dx_1 \wedge \cdots \wedge dx_n. \end{aligned}$$

Hence $d\theta_n = 0$ and θ_n is closed. To prove that θ_n is not exact, let $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the $n-1$ sphere and consider

$$\int_{S^{n-1}} \theta_n = \int_{S^{n-1}} \Theta_n.$$

By the previous part, this is the volume of the unit ball in $\mathbb{R}^n$. This shows that θ_n is not exact, since, otherwise, if $\theta_n = d\omega_n$, then $\int_{S^{n-1}} \theta_n = \int_{\partial S^{n-1}} \omega_n = 0$ since ∂S^{n-1} is empty.

Problem 4. Given linearly independent vectors $\underline{v}_1, ..., \underline{v}_{n-1}$ in $\mathbb{R}^n$, let X be the $n\times(n-1)$ matrix with columns $\underline{v}_1, ..., \underline{v}_{n-1}$ and let X_i be the $(n-1)\times(n-1)$ matrix obtained by omitting row i. Prove that

$$\underline{n} = \frac{1}{\sqrt{\det X^t X}} \sum_{i=1}^{n} (-1)^{i-1} (\det X_i) \underline{e}_i$$

is the unit vector normal to the span of $\underline{v}_1, ..., \underline{v}_{n-1}$ which makes $\underline{n}, \underline{v}_1, ..., \underline{v}_n$ a right handed frame. Hence conclude that the unit normal vector field of an oriented $n-1$ manifold in $\mathbb{R}^n$ is continuous.

Solution. Let $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ and expand by the first column to find

$$\begin{aligned} \det\left(\underline{x}, \underline{v}_1, \cdots, \underline{v}_{n-1}\right) &= \sum_{i=1}^{n} x_i (-1)^{i-1} \det X_i \\ &= \underline{x} \cdot \left(\sum_{i=1}^{n} (-1)^{i-1} (\det X_i) \underline{e}_i \right). \end{aligned}$$

It follows that the determinant is proportional to the component of $\underline{x}$ in the direction $\underline{n}$, and hence that this vector is orthogonal to the span of $\underline{v}_1, ..., \underline{v}_{n-1}$. Since, by Cauchy-Schwarz from lecture,

$$V(X)^2 = \det(X^t X) = \sum_{i=1}^{n} (\det X_i)^2,$$

$\underline{n}$ is a unit vector. Furthermore,

$$\det\left(\underline{n}, \underline{v}_1, \cdots, \underline{v}_{n-1}\right) = V(X) > 0$$

so the frame is right handed.

Problem 5. Given a differential k form ω and a vector $\underline{n}$, define the *contraction* of ω by $\underline{n}$ to be the $k-1$ form $\underline{n}\neg\omega$ given by

$$\underline{n}\neg\omega(\underline{v}_1, ..., \underline{v}_{k-1}) = \omega(\underline{n}, \underline{v}_1, ..., \underline{v}_{k-1}).$$

If $\underline{n} = c_1\underline{e}_1 + \cdots + c_n\underline{e}_n$ calculate $\underline{n}\neg dx_1 \wedge \cdots \wedge dx_n$.

Solution. We have, for $i_1 < i_2 < \cdots < i_{n-1} = (1, 2, ..., \hat{j}, ..., n)$

$$\begin{aligned} &x_1 \wedge \cdots \wedge x_n \left(\sum_{i=1}^{n} c_i \underline{e}_i, \underline{e}_{i_1}, \cdots, \underline{e}_{i_{n-1}} \right) \\ &= c_j x_1 \wedge \cdots \wedge x_n(\underline{e}_j, \underline{e}_1, \underline{e}_2, \cdots, \widehat{\underline{e}_j}, \cdots, \underline{e}_n) \\ &= (-1)^{j-1} c_j. \end{aligned}$$

Hence,

$$\underline{n}\neg dx_1 \wedge \cdots \wedge dx_n = \sum_{j=1}^{n} (-1)^{j-1} c_j dx_1 \wedge \cdots \wedge \widehat{dx}_j \wedge \cdots \wedge dx_n.$$

Problem 6. Let $\omega = (x^2+y^2)dx \wedge dy + e^x dx \wedge dz + e^y dy \wedge dz$. Prove that $d\omega = 0$ and find θ such that $d\theta = \omega$. Then calculate

$$\int_H \omega$$

where H is the upper hemisphere $x^2+y^2+z^2 = 1$, $z \geq 0$ oriented with upward pointing unit normal.

Solution. Let $\theta = \frac{-y^3}{3}dx + \frac{x^3}{3}dy + (e^x + e^y)dz$. Then $d\theta = -y^2 dy \wedge dx + x^2 dx \wedge dy + e^x dx \wedge dz + e^y dy \wedge dz = \omega$. Since ω is exact, it is closed.

Let D be the disc $\{x^2 + y^2 \leq 1, z = 0\}$ oriented by upward pointing unit normal. Then $H - D = \partial M$ where M is the enclosed volume. Since $d\omega = 0$, $\int_M d\omega = 0 = \int_H \omega - \int_D \omega$, so it suffices to calculate $\int_D \omega$. Here dz vanishes, so

$$\int_H \omega = \int_{x^2+y^2\leq 1} (x^2+y^2)dx \wedge dy = \int_0^{2\pi} \int_0^1 r^3 dr d\theta = \frac{\pi}{2}.$$

Problem 7. Let σ be a C^∞ function of compact support on $\mathbb{R}$, $\int_{\mathbb{R}} \sigma = 1$, and define for $t > 0$, $\sigma_t(x) = t\sigma(xt)$. Let f be continuous, of compact support on $\mathbb{R}$. Define the convolution $f * \sigma(x) = \int_y f(y)\sigma(x-y)$. Prove that

$$\frac{d}{dx}(f * \sigma)(x) = f * \sigma'(x)$$

and hence that $f * \sigma$ is C^∞. Furthermore, prove

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |f(x) - f * \sigma_t(x)| = 0$$

and hence that f may be approximated uniformly by C^∞ functions.

Solution. We have

$$\begin{aligned}&\frac{f * \sigma(x+\delta) - f * \sigma(x)}{\delta} - f * \sigma'(x) \\ &= \int_y f(y)\left[\frac{\sigma(x+\delta-y) - \sigma(x-y)}{\delta} - \sigma'(x-y)\right] dy.\end{aligned}$$

By Taylor's theorem, $\left|\frac{\sigma(x+\delta-y)-\sigma(x-y)}{\delta} - \sigma'(x-y)\right| \leq \frac{\delta}{2}\|\sigma''\|_\infty$. Since f is bounded on a compact interval, the limit as $\delta \to 0$ is 0, which proves the first claim. It follows that by taking repeated derivatives this way, $f * \sigma$ is C^∞.

To prove the latter claim, write, using that $\int_{-\infty}^\infty \sigma_t(y)dy = 1$,

$$f * \sigma_t(x) - f(x) = \int_{-\infty}^\infty (f(x-y) - f(x))\sigma_t(y)dy.$$

and, hence, by the triangle inequality,

$$\begin{aligned}|f * \sigma_t(x) - f(x)| &\leq \int_{-\infty}^\infty |f(x-y) - f(x)|\sigma_t(y)dy \\ &\leq \sup\{|f(x-y) - f(x)| : y \in \operatorname{supp} \sigma_t\}.\end{aligned}$$

Since f is continuous on a compact set, it is uniformly continuous there. The claim now follows, since the support of σ_t tends to 0 as $t \to \infty$.

Problem 8. Let M_1 be a compact oriented k manifold without boundary in $\mathbb{R}^m$ and M_2 a compact oriented ℓ manifold without boundary in $\mathbb{R}^n$. Prove that $M_1 \times M_2 = \{(\underline{x}, \underline{y}) : \underline{x} \in M_1, \underline{y} \in M_2\}$ is a compact oriented $k + \ell$ manifold without boundary in $\mathbb{R}^{n+m}$, given the orientation of the product coordinates charts.

Let $\pi_1 : \mathbb{R}^{m+n} \to \mathbb{R}^m$ be projection to the first m coordinates and $\pi_2 : \mathbb{R}^{m+n} \to \mathbb{R}^n$ be projection to the last n coordinates. Let ω be a k form defined on an open set containing M_1 and η be an ℓ form defined on an open set containing M_2. Prove that

$$\int_{M_1 \times M_2} \pi_1^* \omega \wedge \pi_2^* \eta = \int_{M_1} \omega \int_{M_2} \eta.$$

Solution. Let $\alpha : U_1 \to V_1$ be a coordinate chart of M_1 and $\beta : U_2 \to V_2$ be a coordinate chart of M_2, so that U_1 is open in $\mathbb{R}^k$ and U_2 is open in $\mathbb{R}^\ell$. Then $\alpha \times \beta : U_1 \times U_2 \to V_1 \times V_2$ maps an open set of $\mathbb{R}^{k+\ell}$ to $M_1 \times M_2$. Since

$$D(\alpha \times \beta) = \begin{pmatrix} D\alpha & 0 \\ 0 & D\beta \end{pmatrix},$$

$D(\alpha \times \beta)$ has rank $k + \ell$. Also, $\alpha \times \beta$ has the minimum regularity of α and β. Continuity of the inverse function follows from $(\alpha \times \beta)^{-1} = \alpha^{-1} \times \beta^{-1}$. Thus $\alpha \times \beta$ is a coordinate chart, and the collection of charts covering M_1 times those covering M_2 cover $M_1 \times M_2$. If α_1 and α_2 overlap positively with transition function g_{α_1,α_2} and β_1 and β_2 overlap positively with transition function g_{β_1,β_2}, then $\alpha_1 \times \beta_1$ and $\alpha_2 \times \beta_2$ have transition function $g_{\alpha_1,\alpha_2} \times g_{\beta_1,\beta_2}$, and

$$\det D(g_{\alpha_1,\alpha_2} \times g_{\beta_1,\beta_2}) = \det Dg_{\alpha_1,\alpha_2} \times \det Dg_{\beta_1,\beta_2} > 0.$$

Thus $M_1 \times M_2$ is oriented. Since M_1, M_2 are closed and bounded, so is their product, which is compact. The coordinate charts all have open sets in $\mathbb{R}^{k+\ell}$, so $M_1 \times M_2$ does not have a boundary. This proves the first set of claims.

Let $dx_1, ..., dx_m$ be dual to the standard basis vectors in $\mathbb{R}^m$, and $dy_1, ..., dy_n$ dual to the standard basis vectors in $\mathbb{R}^n$. By linearity, it suffices to consider the case that ω and η have support whose intersections with M_1, M_2 is contained in a single coordinate patch. Also, we may assume that $\omega = f_I(x) dx_I$

and $\eta = g_J(y)dy_J$. Then $\pi_1^*\omega \wedge \pi_2^*\eta = f_I(x)g_J(y)dx_I \wedge dy_J$. Thus

$$(\alpha \times \beta)^*(\pi_1^*\omega \wedge \pi_2^*\eta) = (f_I \circ \alpha)(g_J \circ \beta)\det(D\alpha_I)\det(D\beta_J)dz$$

where dz is the volume form on $\mathbb{R}^{k+\ell}$. The product claim now follows by Fubini's theorem.