MATH 319/320, SPRING 2020 PRACTICE MIDTERM 1

FEBRUARY 27

Each problem is worth 10 points.

Problem 1. Prove by induction

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n+1} n^2 = (-1)^{n+1} \frac{n(n+1)}{2}.$$

Solution. Base case ($n = 1$): $1^2 = 1 = (-1)^2 \frac{1 \cdot 2}{2}$.

Inductive step: Assume for some $n \geq 1$ that $1^2 - 2^2 + \cdots + (-1)^{n+1} n^2 = (-1)^{n+1} \frac{n(n+1)}{2}$. Applying the inductive assumption,

$$\begin{aligned}
&1^2 - 2^2 + \cdots + (-1)^{n+1} n^2 + (-1)^{n+2} (n+1)^2 \\
&= (-1)^{n+1} \frac{n(n+1)}{2} + (-1)^{n+2} (n+1)^2 \\
&= (-1)^{n+2} \left[(n+1)^2 - \frac{n(n+1)}{2} \right] \\
&= (-1)^{n+2} \frac{(n+1)(n+2)}{2}.
\end{aligned}$$

This proves the claim by induction.

Problem 2. Let (x_n) be an increasing sequence. Prove that (x_n) converges if and only if it is bounded.

Solution. First suppose that (x_n) is convergent with limit x. Choose N such that $n > N$ implies $|x_n - x| < 1$. Then $|x_n| < |x| + 1$. It follows that for all n, $|x_n| \leq \max(|x_1|, ..., |x_N|, |x| + 1)$, and hence (x_n) is bounded.

Now suppose that (x_n) is bounded. The set $\{x_n\}$ is bounded and non-empty. Let $\alpha = \sup\{x_n\}$. Given $\epsilon > 0$, choose N such that $\alpha - \epsilon < x_N \leq \alpha$. For $n > N$,

$$\alpha - \epsilon < x_N \leq x_n \leq \alpha$$

and hence $|\alpha - x_n| < \epsilon$. Thus $\lim x_n = \alpha$.

Problem 3. Prove that for all positive real numbers $x > 0$ there is an integer n such that $0 < \frac{1}{n} < x$.

Solution. If there exists a natural number $n > \frac{1}{x}$, then $0 < \frac{1}{n} < x$, so it suffices to prove that the natural numbers do not have an upper bound. Suppose to the contrary that $\frac{1}{x}$ is an upper bound for $\mathbb{N}$, and let α be a least upper bound. Then $\alpha - 1$ is not an upper bound, so there exists natural number $m > \alpha - 1$. It follows that $m + 1 > \alpha$, contradiction.

Problem 4. State carefully the definition of the supremum of a bounded, non-empty set S of real numbers. Prove that $\sup S = -\inf(-S)$, where $-S = \{-s : s \in S\}$.

Solution. The supremum of a bounded non-empty set is an upper bound for the set such that any other upper bound is at least as large.

Let $\alpha = \sup S$. Then α is an upper bound, so that, for any $s \in S$, $s \leq \alpha$. It follows that $-s \geq -\alpha$, so $-\alpha$ is a lower bound for $-S$. Since α is the least upper bound, for any $\epsilon > 0$, $\alpha - \epsilon$ is not an upper bound, so that there exists $s \in S$ with $s > \alpha - \epsilon$. Then $-\alpha + \epsilon > -s$ so $-\alpha + \epsilon$ is not a lower bound for $-S$. It follows that $-\alpha$ is the greatest lower bound for $-S$, as claimed.

Problem 5. Show that there exists a positive real number x such that $x^3 = 2$. Prove that x is irrational.

Solution. Let $S = \{s \in \mathbb{R} : s > 0, s^3 < 2\}$. Then $1 \in S$ so S is non-empty. If $s > 2$ then $s^3 \geq 2^3 = 8$, $s \notin S$. Thus S is bounded above. Let $\alpha = \sup S$.

First, suppose that $\alpha^3 > 2$. In particular, $\alpha > 1$. Let $0 < \epsilon < 1$ be a small number. Then

$$\begin{aligned}(\alpha - \epsilon)^3 &= \alpha^3 - 3\alpha^2\epsilon + 3\alpha\epsilon^2 - \epsilon^3 \\ &> \alpha^3 - 3\alpha^2\epsilon - \epsilon^3 \\ &\geq \alpha^3 - (3\alpha^2 + 1)\epsilon.\end{aligned}$$

Set $\epsilon = \min\left(\frac{1}{2}, \frac{\alpha^3-2}{3\alpha^2+1}\right)$. Then $\alpha - \epsilon > 0$ and $(\alpha - \epsilon)^3 > 2$, so $\alpha - \epsilon$ is still an upper bound, and α is not a least upper bound.

Suppose instead that $\alpha^3 < 2$. Let $0 < \epsilon < 1$ and note

$$(\alpha + \epsilon)^3 = \alpha^3 + 3\alpha^2\epsilon + 3\alpha\epsilon^2 + \epsilon^3 < \alpha^3 + (3\alpha^2 + 3\alpha + 1)\epsilon.$$

Choose $\epsilon = \min\left(\frac{1}{2}, \frac{2-\alpha^3}{3\alpha^2+3\alpha+1}\right)$. Then $(\alpha + \epsilon)^3 < 2$, so $\alpha + \epsilon \in S$, which contradicts α is an upper bound.

By trichotomy, the only possibility left is $\alpha^3 = 2$.

To check α irrational, suppose instead $\alpha = \frac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$, and having greatest common divisor 1. Then $p^3 = 2q^3$ implies 2 divides p, so $p = 2p'$ and $p^3 = 8(p')^3$. Then $4(p')^3 = q^3$ implies 2 divides q, a contradiction. Thus α is irrational.