# SPRING 2020: MAT 320 MIDTERM 2

The purpose of this midterm exam is to prove a version of the existence and uniqueness theorem for ordinary differential equations. Solve the problems independently without outside aids. Your solutions should be submitted on Blackboard as for the weekly homework assignments.

**Problem 1.** (20 pts) Suppose $f$ is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number $A$ such that $|f'(x)| \le A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Hint: Fix $x_0 \in [a, b]$, let

$$M_0 = \sup |f(x)|, \qquad M_1 = \sup |f'(x)|$$

for $a \le x \le x_0$. For any such $x$,

$$|f(x)| \le M_1(x_0 - a) \le A(x_0 - a)M_0.$$

Thus $M_0 = 0$ if $A(x_0 - a) < 1$.

**Problem 2.** (30 pts) Let $\phi$ be a real function defined on the rectangle $R$ in the plane, given by $a \le x \le b$, $\alpha \le y \le \beta$. A *solution* of the initial-value problem

$$y' = \phi(x, y), \qquad y(a) = c \qquad (\alpha \le c \le \beta)$$

is, by definition, a differentiable function $f$ on $[a, b]$ such that $f(a) = c$, $\alpha \le f(x) \le \beta$, and

$$f'(x) = \phi(x, f(x)) \qquad (a \le x \le b).$$

Prove that such a problem has at most one solution if there is a constant $A$ such that

$$|\phi(x, y_2) - \phi(x, y_1)| \le A|y_2 - y_1|$$

whenever $(x, y_1), (x, y_2) \in R$.

The following proof of the existence theorem for ODE's uses several concepts for functions of two variables which are generalizations of concepts for functions of one variable.

**Definition 1.** *A function $f : [a, b] \times [c, d] \to \mathbb{R}$ is continuous at $(x_0, y_0)$ if, for any $\epsilon > 0$ there exists $\delta > 0$ such that, if $\|(x - x_0, y - y_0)\|_2 < \delta$ then $|f(x, y) - f(x_0, y_0)| < \epsilon$. A function $f$ is uniformly continuous if, for any $\epsilon > 0$ there exists $\delta > 0$ such that, whenever $\|(x_1 - x_2, y_1 - y_2)\|_2 < \delta$, then $|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$.*

You may use the fact that a continuous function on $[a, b] \times [c, d]$ is uniformly continuous. For a proof of this fact, see the taped version of Lecture 18.

**Problem 3.** (50 pts) Suppose $\phi$ is a continuous bounded real function in the strip defined by $0 \leq x \leq 1$, $-\infty < y < \infty$. Prove that the initial-value problem

$$y' = \phi(x, y), \qquad y(0) = c$$

has a solution. It may help to follow the following steps.

a. For each $n$, let $x_i = \frac{i}{n}$, $i = 0, 1, 2, ..., n$ and define a piecewise linear continuous function $f_n(t)$ by

$$f_n'(t) = \phi(x_i, f_n(x_i)), \qquad \text{if } \ x_i < t < x_{i+1}.$$

Define $\Delta_n(t) = f_n'(t) - \phi(t, f_n(t))$ except at $t = x_i$, where we define $\Delta_n(x_i) = 0$. Explain why

$$f_n(x) = c + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)]dt.$$

b. Let $M$ be such that $|\phi(x, y)| \leq M$. Prove that $|f_n'(x)| \leq M$, $|\Delta_n(x)| \leq 2M$, $\Delta_n$ is Riemann integrable, and $|f_n(x)| \leq M + |c|$ for all $n$ and $x \in (0, 1)$.

c. Show that there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(q)$ converges to a value $f(q)$ for all rational $q \in [0, 1]$. (Hint: enumerate the rationals in $[0, 1]$ as $q_1, q_2, q_3, ....$ First find a subsequence $f_{n_k}^{(1)}(q_1)$ which converges, which is possible by the Bolzano-Weistrass theorem. Next find a subsequence $f_{n_k}^{(2)}(q_2)$ which is a sub-subsequence of $f_{n_k}^{(1)}$, and which converges. Iterate this. As the final subsequence, choose $f_{n_k}^{(k)}$ and check that this converges at each rational.)

d. Check that the subsequence $\{f_{n_k}\}$ which you found in part c. converges uniformly to a continuous function $f(x)$ on $[0, 1]$, in the sense that, as $k \to \infty$, $\sup_{x \in [0,1]} |f(x) - f_{n_k}(x)| \to 0$. It will be helpful to use the bound $|f_{n_k}'(x)| \leq M$ for all $x$ and all $k$.

e. Check that $\phi$ is uniformly continuous on $0 \leq x \leq 1$, $|y| \leq M + |c|$, and hence

$$\phi(t, f_{n_k}(t)) \to \phi(t, f(t))$$

uniformly on $[0, 1]$.

f. Check that $\Delta_n(t)$ converges to 0 uniformly on $[0, 1]$, since

$$\Delta_n(t) = \phi(x_i, f_n(x_i)) - \phi(t, f_n(t))$$

in $(x_i, x_{i+1})$.

g. Hence

$$f(x) = c + \int_0^x \phi(t, f(t))dt.$$

Thus $f(x)$ is a solution to the given problem.