MATH 319/320, SPRING 2020 MIDTERM 1

FEBRUARY 27

Each problem is worth 10 points.

Problem 1. Let $F_0 = F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ be the Fibonacci sequence. Prove that for all natural numbers n, $F_n \leq 2^n$.

Solution. We prove this by strong induction.

Base case ($n = 1, 2$): $F_1 = 1 < 2^1 = 2$. $F_2 = 2 < 2^2 = 4$.

Inductive step: Assume for some $n \geq 2$ that for all $1 \leq k \leq n$, $F_k \leq 2^k$. Then $F_{n+1} = F_n + F_{n-1} \leq 2^n + 2^{n-1} < 2^n + 2^n = 2^{n+1}$ as wanted.

Problem 2. Prove that $x^2 = 3$ does not have a rational solution, but that it has a positive real solution.

Solution. Suppose $x^2 = 3$ has a rational solution $\frac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$ and with gcd 1. Then $p^2 = 3q^2$ implies $3|p$. Let $p = 3p'$. Then $3(p')^2 = q^2$ implies 3 divides q, a contradiction.

To prove that there is a positive real solution, let $S = \{x \in \mathbb{R} : x > 0, x^2 < 3\}$. Then $1 \in S$ so S is non-empty. Also, if $x \geq 3$ then $x^2 \geq 3$ so S is bounded by 3. Let $\alpha = \sup S$. Note that $\alpha > 1$.

Suppose first that $\alpha^2 > 3$. Let $0 < \epsilon < 1$. Then $(\alpha - \epsilon)^2 = \alpha^2 - 2\alpha\epsilon + \epsilon^2 > \alpha^2 - 2\alpha\epsilon$. Let $\epsilon = \min\left(\frac{1}{2}, \frac{\alpha^2-3}{2\alpha}\right)$. It follows that $\alpha - \epsilon > 0$ and $(\alpha - \epsilon)^2 > 3$, so $\alpha - \epsilon$ is an upper bound for S, contradiction.

Next suppose that $\alpha^2 < 3$. Let $0 < \epsilon < 1$. Then $(\alpha + \epsilon)^2 = \alpha^2 + 2\alpha\epsilon + \epsilon^2 < \alpha^2 + (2\alpha + 1)\epsilon$. Choose $\epsilon = \min\left(\frac{1}{2}, \frac{3-\alpha^2}{2\alpha+1}\right)$. Then $\alpha + \epsilon > 0$ and $(\alpha + \epsilon)^2 < 3$ so α is not an upper bound for S.

Hence, by trichotomy, $\alpha^2 = 3$.

Problem 3. Prove that each non-empty open interval of $\mathbb{R}$ contains both rational and irrational numbers.

Solution. Let $a < b$. We first check that there is a rational number r with $a < r < b$. If $a < 0 < b$ then we are already done, so we may assume a and b have the same sign, which may be assumed positive, or take negatives. Choose an integer $n > \frac{1}{b-a}$ so that $na + 1 < nb$. Let m be the least integer greater than na. Then $m-1 \leq na$ implies $m < nb$. It follows that $a < \frac{m}{n} < b$.

To prove the claim for irrationals, assume again that a and b have the same sign, since otherwise $a < 0 < b$ and we can replace a with $\frac{b}{2}$. Assume $0 < a < b$. Find rational r with $\frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}$. Then $a < r\sqrt{2} < b$ and $r\sqrt{2}$ is irrational, since otherwise $\sqrt{2}$ would be rational.

Problem 4. Show that if $z_n = (a^n + b^n)^{\frac{1}{n}}$ where $0 < a < b$, then $\lim(z_n) = b$.

Solution. Write $z_n = b\left(1 + \left(\frac{a}{b}\right)^n\right)^{\frac{1}{n}}$. We show that, for every $\epsilon > 0$, for all sufficiently large n, $1 \leq \left(1 + \left(\frac{a}{b}\right)^n\right)^{\frac{1}{n}} \leq 1 + \epsilon$, which implies the claim. Since $0 < \frac{a}{b} < 1$, $\left(\frac{a}{b}\right)^n$ is decreasing and converges to 0. Thus, for all n sufficiently large,

$$1 \leq 1 + \left(\frac{a}{b}\right)^n \leq 1 + \epsilon < 1 + n\epsilon \leq (1 + \epsilon)^n.$$

Taking nth roots proves the claim.

Problem 5. Let $S \subset \mathbb{R}$ be a bounded set and $S_0 \subset S$ a non-empty subset. Prove

$$\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S.$$

Solution. Let $\alpha = \inf S$. Then α is a lower bound for S so, for any $s \in S_0$, $s \in S$ so that $\alpha \leq s$. In particular, $\alpha \leq \inf S_0$. Let $\beta = \sup S$. Then β is an upper bound for S, so, for any $s \in S_0$, $s \in S$, $s \leq \beta$. Thus β is an upper bound for S_0 so $\sup S_0 < \beta$. Let $s \in S_0$. Then $\inf S_0 \leq s \leq \sup S_0$. Chaining together these inequalities proves the claim.