SPRING 2020: MAT 320 FINAL EXAM

The purpose of this final exam is to prove two theorems on the uniform approximation of continuous functions with functions from a family. The Weierstrass approximation theorem approximates a continuous function on an interval uniformly with polynomials. The Cesàro means of a function's Fourier series approximate the function uniformly with wave functions of bounded frequency. The proofs illustrate the convolution kernel method of approximating a function with its convolution with a kernel approximating the identity. Solve the problems independently without outside aids. Your solutions should be submitted on Blackboard as for the weekly homework assignments.

Problem 1 (Weierstrass approximation theorem)**.** *(25 pts) Prove the following: If f is a continuous function on $[a, b]$, there exists a sequence of polynomials P_n converging uniformly to f on $[a, b]$. It may help to follow the following steps.*

a. *Since $P(a + (b - a)t)$ is a polynomial of $t \in [0, 1]$ if P is a polynomial, we may assume that $[a, b] = [0, 1]$. Also, it suffices to assume that $f(0) = f(1) = 0$, since $g(x) = f(x) - f(0) - x(f(1) - f(0))$ differs from $f(x)$ by a polynomial and satisfies $g(0) = g(1) = 0$. Extend f to a function on $\mathbb{R}$ as $f(x) = 0$ if $x \notin [0, 1]$.*
b. *Let $Q_n(x)$ be the polynomial $Q_n(x) = c_n(1 - x^2)^n$, where the constant c_n is chosen so that $\int_{-1}^{1} Q_n(x)dx = 1$. Prove that there is a constant $c > 0$ such that $c_n \leqslant c\sqrt{n}$.*
c. *Prove that for each fixed $\delta > 0$, as $n \to \infty$, $\int_{-\delta}^{\delta} Q_n(x)dx \to 1$.*
d. *Define $P_n(x) = \int_{-1}^{1} f(x - t)Q_n(t)dt = \int_{x-1}^{x+1} f(t)Q_n(x - t)dt$. Show that $P_n(x)$ is a polynomial.*
e. *Write $f(x) - P_n(x) = \int_{-1}^{1}(f(x) - f(x-t))Q_n(t)dt$ and prove that $|f(x) - P_n(x)|$ tends to 0 uniformly in x as $n \to \infty$ by splitting the integral into the parts where $|t| < \delta$ for some fixed $\delta > 0$ and the remainder.*

We say a function $f : \mathbb{R} \to \mathbb{R}$ is 1-periodic if $f(x + 1) = f(x)$ and write $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$. If f is Riemann integrable on $[0, 1]$, its nth Fourier coefficient is $\hat{f}(n) = \int_0^1 f(x)e^{-2\pi i n x}dx$. The Fourier series of f is the (possibly divergent) sum

$$Sf(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi i n x}.$$

It is known that if f is continuous, the sequence of partial sums $\sum_{-N}^{N} \hat{f}(n)e^{2\pi i n x}$ may not converge to $f(x)$ as $N \to \infty$. Fejér's Theorem shows that the Cesàro averages of these sums converge to f uniformly.

Problem 2 (Fejér's Theorem). *(20 pts) Given a continuous function $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$, prove that the Cesàro averages of the partial sums of the Fourier series,*

$$\begin{aligned}\sigma_N(f)(x) &= \frac{1}{N+1} \sum_{M=0}^{N} \sum_{|k| \leqslant M} \hat{f}(k) e^{2\pi i k x} \\ &= \sum_{|k| \leqslant N} \left(1 - \frac{|k|}{N+1}\right) \hat{f}(k) e^{2\pi i k x}\end{aligned}$$

converge uniformly to f. It may help to follow the following steps.

a. *Define the Fejér kernel*

$$\mathscr{F}_N(x) = \sum_{|k| \leqslant N} \left(1 - \frac{|k|}{N+1}\right) e^{2\pi i k x}.$$

Prove that $\int_0^1 \mathscr{F}_N(x)dx = 1$ and, for $x \neq 0 \bmod 1$,

$$\mathscr{F}_N(x) = \frac{1}{N+1} \left| \frac{e^{2\pi i (N+1) x} - 1}{e^{2\pi i x} - 1} \right|^2 = \frac{1}{N+1} \left(\frac{\sin(N+1)\pi x}{\sin \pi x} \right)^2.$$

b. *Prove that, for each $\delta > 0$, $\int_{-\delta}^{\delta} \mathscr{F}_N(x)dx \to 1$ as $N \to \infty$.*

c. *Prove that*

$$\sum_{|k| \leqslant N} \left(1 - \frac{|k|}{N+1}\right) \hat{f}(k) e^{2\pi i k x} = \int_0^1 \mathscr{F}_N(x-t) f(t) dt.$$

d. *Using that*

$$f(x) - \sigma_N(f)(x) = \int_0^1 \mathscr{F}_N(x-t)(f(x) - f(t))dt,$$

prove that $\sigma_N(f)$ converges uniformly to f on $[0,1]$.

Problem 3. *(10 pts) Let $a_0, a_1, a_2, \ldots$ be a convergent sequence, converging to a. Prove that the Cesàro averages $\sigma_N = \frac{1}{N+1}(a_0 + \cdots + a_N) \to a$ as $N \to \infty$.*

Problem 4. *(20 pts) Calculate the Fourier series of the 1-periodic function f which is defined on $[0,1]$ by*

$$f(x) = \begin{cases} x & 0 \leqslant x \leqslant \frac{1}{2} \\ 1 - x & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}.$$

Evaluate the Fourier series at 0 to check that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.