MATH 319/320, FALL 2022 MIDTERM 1

SEPTEMBER 22

Each problem is worth 10 points.

Problem 1. Let the power set of a set S be $P(S) = \{A : A \subset S\}$, the set of all subsets of S. Prove by induction that if S has $n \geq 1$ elements, then $P(S)$ has 2^n elements.

Solution. Base case ($n = 1$): If $|S| = 1$ the power set is $\{\emptyset, S\}$.

Inductive step: Let $n \geq 1$ and assume the claim holds for all sets of size n. Let S be a set of size $n + 1$ and let $x \in S$. Split the power set of S into subsets that contain x and subsets that do not. Those that do not form the power set of $S \setminus \{x\}$, and hence there are 2^n of these. Those that contain x can be found by appending x to each subset in the power set of $S \setminus \{x\}$, for another 2^n subsets. Thus there are 2^{n+1} elements in the power set of S.

Problem 2. Let $K := \{s + t\sqrt{2} : s, t \in \mathbb{Q}\}$. Show that K satisfies the following

a. If $x_1, x_2 \in K$, then $x_1 + x_2 \in K$ and $x_1 x_2 \in K$.
b. If $x \neq 0$ and $x \in K$, then $\frac{1}{x} \in K$.

Solution. a. Let $x_1 = (s_1 + t_1\sqrt{2})$ and $x_2 = (s_2 + t_2\sqrt{2})$, $s_1, s_2, t_1, t_2 \in \mathbb{Q}$. Then $x_1 + x_2 = (s_1 + s_2) + (t_1 + t_2)\sqrt{2}$ shows $x_1 + x_2 \in K$. Also $x_1 x_2 = (s_1 + t_1\sqrt{2})(s_2 + t_2\sqrt{2}) = (s_1 s_2 + 2t_1 t_2) + (s_1 t_2 + s_2 t_1)\sqrt{2}$ shows $x_1 x_2 \in K$.

b. Let $x = s + t\sqrt{2}$ with not both $s, t = 0$. Then $\frac{1}{x} = \frac{s - t\sqrt{2}}{s^2 - 2t^2}$. Notice the denominator does not vanish since 2 is not a square. Since $\frac{s}{s^2 - 2t^2}$ and $\frac{t}{s^2 - 2t^2}$ are both rational, this completes the proof.

Problem 3. Calculate the following limits.

a. Using only the definition of limits, calculate the limit $\lim \frac{4n^2+3}{2n^2+1}$.

b. If $0 < a < b$, determine $\lim \left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)$. (You do not have to use the definition and you may rely on properties of the limit.)

Solution. a. Write $\frac{4n^2+3}{2n^2+1} = 2 + \frac{1}{2n^2+1}$. We show the limit is 2. We need to show that for $\epsilon > 0$ we can find N so that $n > N$ implies $\frac{1}{2n^2+1} < \epsilon$. It suffices to choose $N = \frac{1}{\sqrt{\epsilon}}$, which implies

$$\left|\frac{4n^2+3}{2n^2+1} - 2\right| = \frac{1}{2n^2+1} < \frac{1}{\frac{2}{\epsilon}+1} < \frac{\epsilon}{2}.$$

b. Write $\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = b\frac{1+\left(\frac{a}{b}\right)^{n+1}}{1+\left(\frac{a}{b}\right)^n}$. Thus

$$\lim \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = b\frac{1+\lim\left(\frac{a}{b}\right)^{n+1}}{1+\lim\left(\frac{a}{b}\right)^n} = b.$$

Problem 4. Establish the convergence or divergence of the sequence (y_n) where

$$y_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, \qquad n \in \mathbb{N}.$$

Solution. We have $y_{n+1} - y_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0$. Thus the sequence y_n is increasing. It is also bounded above by 1 since each of the n terms in the sum defining y_n has size at most $\frac{1}{n+1}$. Thus it converges.

Problem 5. Show that if $x_n > 0$ for all $n \in \mathbb{N}$, then $\lim x_n = 0$ if and only if $\lim \frac{1}{x_n} = \infty$.

Solution. Suppose $\lim x_n = 0$. Given $M > 0$ choose N so that $n > N$ implies $x_n < \frac{1}{M}$ so $\frac{1}{x_n} > M$ and this proves $\lim \frac{1}{x_n} = \infty$. Conversely if $\lim \frac{1}{x_n} = \infty$, given $\epsilon > 0$ choose N so that $n > N$ implies $\frac{1}{x_n} > \frac{1}{\epsilon}$. Then $0 < x_n < \epsilon$ and so $\lim x_n = 0$.