MATH 320, FALL 2017 FINAL EXAM

DECEMBER 15

Each problem is worth 10 points.

Problem 1. Let $\{f_n\}$ be a sequence of continuous functions on an interval $[a, b]$ converging uniformly to a function f on $[a, b]$.

a. (5 points) Prove that f is continuous.

b. (5 points) Prove that $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$ as $n \to \infty$.

Solution.

a. We check continuity of f at $x_0 \in [a, b]$. Given $\epsilon > 0$, let N be such that $n > N$ implies, for all $x \in [a, b]$, $|f_N(x) - f(x)| < \frac{\epsilon}{3}$. Since f_{N+1} is continuous at x_0, let $\delta > 0$ be such that $x \in [a, b]$ and $|x - x_0| < \delta$ then $|f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\epsilon}{3}$. It follow that, by the triangle inequality,

$$|f(x) - f(x_0)|$$
$$\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(x_0)| + |f_{N+1}(x_0) - f(x_0)|$$
$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

b. By part a, f is integrable. Given $\epsilon > 0$ let N be such that $n > N$ implies for all $x \in [a, b]$, $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$. Then

$$\int_a^b f_n(x) - \frac{\epsilon}{b-a} dx < \int_a^b f(x)dx < \int_a^b f_n(x) + \frac{\epsilon}{b-a} dx$$

so

$$\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| < \epsilon,$$

which proves the limit.

Problem 2.

a. (4 points) State the definition of a real function f differentiable at a point a.

b. (6 points) A function f is Lipschitz with Lipschitz constant M on an interval $[a, b]$ if, for any $x, y \in [a, b]$,

$$|f(x) - f(y)| \leq M|x - y|.$$

Suppose that f is differentiable on (a, b) and $|f'(x)| \leq M$ for all $x \in (a, b)$. Prove that f is Lipschitz on $[a, b]$ with Lipschitz constant M.

Solution.

a. A function f is differentiable at a point $a \in \mathbb{R}$ if it is defined in an open interval containing a and if the limit $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists and is finite.

b. Given $a \leq x < y \leq b$, the Mean Value Theorem states that there is $z \in (x, y)$ such that

$$f'(z) = \frac{f(y) - f(x)}{y - x}.$$

Hence $|f(y) - f(x)| = |y - x||f'(z)| \leq M|y - x|$, so f is Lipschitz with constant M.

Problem 3.

a. (4 points) State the integral form of Taylor's theorem with remainder.

b. (6 points) Determine the Taylor series of $\cos x$ about 0 and prove that the degree k Taylor polynomial of $\cos x$ differs from $\cos x$ by at most $\frac{|x|^{k+1}}{(k+1)!}$.

Solution.

a. Let $n \geq 1$ and let f be n times continuously differentiable on the interval (a, b) and let $c, x \in (a, b)$. Then

$$f(x) = \sum_{j=0}^{n-1} \frac{f^{(j)}(c)(x-c)^j}{j!} + \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt.$$

b. Since $\frac{d}{dx} \cos x = -\sin x$ and $\frac{d}{dx} - \sin x = -\cos x$, the derivatives of $\cos x$ at 0 are given by $\left(\frac{d}{dx}\right)^n \cos x$ is 0 if n is odd, 1 if n is divisible by 4, and -1 if n is even but not divisible by 4. Hence the Taylor series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

The remainder term from the degree k Taylor polynomial is

$$R_k(x) = \int_0^x \frac{(x-t)^k}{k!} \left(\frac{d}{dt}\right)^{k+1} (\cos t)dt.$$

Since the derivatives of $\cos x$ are bounded in size by 1, the triangle inequality for integrals implies that

$$|R_k(x)| \leq \int_0^{|x|} \frac{(|x|-t)^k}{k!} dt = \frac{|x|^{k+1}}{(k+1)!}.$$

Problem 4.

a. (5 points) Let f be real valued and increasing on an interval $[a, b]$. Prove that f is integrable on $[a, b]$.

b. (5 points) Suppose f is continuous on $[a, b]$ and that $\int_a^b f^2(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution.

a. Let $N \geq 1$ and take the partition P_N which divides $[a, b]$ into N equal segments. Write $L(f, N)$ for the lower Darboux sum and $U(f, N)$ for the upper Darboux sum. Then since f is increasing

$$L(f, N) = \frac{b-a}{N} \sum_{j=1}^{N} f\left(a + (j-1)\frac{b-a}{N}\right),$$

$$U(f, N) = \frac{b-a}{N} \sum_{j=1}^{N} f\left(a + j\frac{b-a}{N}\right),$$

so

$$U(f, N) - L(f, N) = \frac{b-a}{N}(f(b) - f(a)).$$

It follows that $U(f, N) - L(f, N) \to 0$ as $N \to \infty$, so the lower and upper Darboux integrals are equal.

b. Suppose $f(x_0) \neq 0$ for some $x_0 \in (a, b)$. Then, by continuity, there is $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ and $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \frac{|f(x_0)|}{2}$. From the triangle inequality, it follows that, for $|x - x_0| < \delta$

$$|f(x)| \geq |f(x_0)| - |f(x_0) - f(x)| > \frac{|f(x_0)|}{2}.$$

Let $g(x) = \frac{f(x_0)^2}{4}$ for $|x - x_0| < \delta$, $g(x) = 0$ otherwise. Then $f(x)^2 > g(x)$ for all x, and hence

$$\int_a^b f(x)^2 dx > \int_a^b g(x)dx = 2\delta\frac{f(x_0)^2}{4} > 0.$$

Problem 5. Determine the following limits.

a. (4 points)

$$\lim_{x\to\infty} xe^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{t^2}{2}}dt.$$

b. (6 points)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1}\left(1+\frac{1}{n}\right)^j.$$

Solution.

a. Bound

$$\int_x^\infty e^{-\frac{t^2}{2}}dt = \int_0^\infty e^{-\frac{(x+t)^2}{2}}dt \le e^{-\frac{x^2}{2}}\int_0^\infty e^{-tx}dt \le \frac{1}{xe^{\frac{x^2}{2}}}.$$

Thus if the limit is written as

$$\lim_{x\to\infty} \frac{\int_x^\infty e^{-\frac{t^2}{2}}dt}{\frac{1}{x}e^{-\frac{x^2}{2}}}$$

then the limit is indeterminant of type $\frac{0}{0}$. It follows on applying l'Hospital's rule and the Fundamental Theorem of Calculus that the limit is equal to

$$\lim_{x\to\infty} \frac{e^{-\frac{x^2}{2}}}{\left(1+\frac{1}{x^2}\right)e^{-\frac{x^2}{2}}} = 1.$$

b. The sum is a geometric series, equal to

$$\frac{\left(1+\frac{1}{n}\right)^n - 1}{\frac{1}{n}}$$

and hence the limit is $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n - 1$. Calculate

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \exp\left(\lim_{n\to\infty} n\log\left(1+\frac{1}{n}\right)\right).$$

If $f(x) = \log(1+x)$ then since $f(1) = 0$, the internal limit is $f'(0) = 1$. Putting together these calculations obtains

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(1 + \frac{1}{n}\right)^j = e - 1.$$

Problem 6.

a. (5 points) Determine the Taylor series of $\log(1+x)$ about $x=0$, and determine the radius of convergence.
b. (5 points) Find the degree 4 Taylor polynomial of $\frac{2}{e^x+e^{-x}}$ about $x=0$.

Solution.

a. The geometric series $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$ has radius of convergence 1. Hence, by the theorem on integration of power series,

$$\log(1+x)=\int_0^x \frac{dt}{1+t}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}$$

has radius of convergence 1.

b. Since $\frac{d}{dx}e^x=e^x$, the Taylor expansion of $f(x)=\frac{e^x+e^{-x}}{2}$ about $x=0$ is $1+\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)$. Let the degree 4 Taylor expansion of $\frac{2}{e^x+e^{-x}}$ about $x=0$ be given by

$$c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+O(x^5).$$

Note that, since $\frac{1}{f(x)}$ is even, $c_1=c_3=0$. Calculate formally,

$$(c_0+c_2x^2+c_4x^4+O(x^5))\left(1+\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)=1+O(x^5)$$

which implies that $c_0=1$, $c_2=-\frac{1}{2}$ and $c_4-\frac{1}{4}+\frac{1}{24}=0$ so $c_4=\frac{5}{24}$. The rigorous justification of this formal calculation is that $f(x)^{n+1}\left(\frac{d}{dx}\right)^n\frac{1}{f(x)}$ is a polynomial in $(f(x),f'(x),...,f^{(n)}(x))$, so that the answer would be unchanged if $f(x)$ were in fact a polynomial. In this case, $\frac{1}{f(x)}$ may be expanded in partial fractions and it's Taylor expansion has a positive radius of convergence.