MATH 320, FALL 2017 FINAL EXAM

DECEMBER 15

Each problem is worth 10 points.

Problem 1. Let $\{f_n\}$ be a sequence of continuous functions on an interval $[a, b]$ converging uniformly to a function f on $[a, b]$.

a. (5 points) Prove that f is continuous.

b. (5 points) Prove that $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$ as $n \to \infty$.

Problem 2.

a. (4 points) State the definition of a real function f differentiable at a point a.

b. (6 points) A function f is Lipschitz with Lipschitz constant M on an interval $[a, b]$ if, for any $x, y \in [a, b]$,

$$|f(x) - f(y)| \leq M|x - y|.$$

Suppose that f is differentiable on (a, b) and $|f'(x)| \leq M$ for all $x \in (a, b)$. Prove that f is Lipschitz on $[a, b]$ with Lipschitz constant M.

Problem 3.

a. (4 points) State the integral form of Taylor's theorem with remainder.

b. (6 points) Determine the Taylor series of $\cos x$ about 0 and prove that the degree k Taylor polynomial of $\cos x$ differs from $\cos x$ by at most $\frac{|x|^{k+1}}{(k+1)!}$.

Problem 4.

a. (5 points) Let f be real valued and increasing on an interval $[a, b]$. Prove that f is integrable on $[a, b]$.

b. (5 points) Given a partition $P = \{a = t_0 < t_1 < \cdots < t_k = b\}$ of $[a, b]$ and a function f on $[a, b]$, the variation of f along P, written $\text{var}(f, P)$ is $\sum_{j=1}^{k} |f(t_j) - f(t_{j-1})|$. Prove that if a partition Q refines P then $\text{var}(f, Q) \geq \text{var}(f, P)$.

Remark: The variation of f on $[a, b]$ is defined to be the sup of $\text{var}(f, P)$ over all partitions P of $[a, b]$. It may be shown that a function of bounded variation is the difference of two increasing functions, and hence is integrable. You do not need to prove this.

Problem 5. Determine the following limits.

a. (4 points)

$$\lim_{x\to\infty} xe^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{t^2}{2}}\,dt.$$

b. (6 points)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1}\left(1+\frac{1}{n}\right)^j.$$

Problem 6.

a. (5 points) Determine the Taylor series of $\log(1+x)$ about $x = 0$, and determine the radius of convergence.

b. (5 points) Find the degree 4 Taylor polynomial of $\frac{2}{e^x+e^{-x}}$ about $x = 0$.