MATH 320, FALL 2017 MIDTERM 2

NOVEMBER 7

Each problem is worth 10 points.

Problem 1.

a. (3 points) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Define

$$\limsup_{n\to\infty} a_n.$$

b. (7 points) Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences of real numbers. Assume that $\limsup a_n$ and $\limsup b_n$ are finite. Prove that

$$\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Give an example where equality does not hold.

Solution.

a. Define for integer $N \ge 1$, $s_N = \sup\{a_n : n \ge N\}$. If $s_N = \infty$ for all N then $\limsup_{n\to\infty} a_n = \infty$. Otherwise, $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} s_N$.

b. For integer $N \ge 1$, let

$$s_N = \sup\{a_n : n \ge N\}, \qquad t_N = \sup\{b_n : n \ge N\},$$

and assume that N is sufficiently large so that both of these suprema are finite. Since s_N is an upper bound for $\{a_n : n \ge N\}$ and t_N is an upper bound for $\{b_n : n \ge N\}$, $s_N + t_N$ is an upper bound for $\{a_n + b_n : n \ge N\}$, so

$$r_N = \sup\{a_n + b_n : n \ge N\}$$

satisfies $r_N \le s_N + t_N$. Hence

$$\begin{aligned}\limsup(a_n+b_n) &= \lim_{N\to\infty} r_N \\ &\le \lim_{N\to\infty}(s_N+t_N) = \limsup a_n + \limsup b_n.\end{aligned}$$

An example in which equality does not hold is

$$a_n = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}, \qquad b_n = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}.$$

Then $\limsup a_n = \limsup b_n = 1$, while $\limsup(a_n + b_n) = 1$.

Problem 2.

a. (3 points) State the definition of a metric d on a set S.

b. (7 points) Given two points $\underline{x} = (x_1, ..., x_n)$ and $\underline{y} = (y_1, ..., y_n)$ in $\mathbb{R}^n$, the ℓ^1 and ℓ^∞ distances between $\underline{x}$ and $\underline{y}$ are

$$d_1(\underline{x}, \underline{y}) = \sum_{i=1}^{n} |x_i - y_i|, \qquad d_\infty(\underline{x}, \underline{y}) = \max\{|x_i - y_i|, i = 1, ..., n\}.$$

Check that the ℓ^1 and ℓ^∞ distances are metrics on $\mathbb{R}^n$, then check that a sequence $\{\underline{x}_k\}_{k\in\mathbb{N}}$ of elements of $\mathbb{R}^n$ converges in the ℓ^1 metric if and only if it converges in the ℓ^∞ metric.

Solution.

a. A metric d is a function $d : S \times S \to \mathbb{R}_{\geq 0}$ satisfying
   i. For all $x \in S$, $d(x, x) = 0$, and for all $x \neq y$ in S, $d(x, y) > 0$.
   ii. For all x, y in S, $d(x, y) = d(y, x)$.
   iii. The triangle inequality holds: for all x, y, z in S, $d(x, z) \leq d(x, y) + d(y, z)$.

b. ℓ^1 metric:
   i. By non-negativity of the absolute value,

$$d_1(\underline{x}, \underline{y}) = \sum_{i=1}^{n} |x_i - y_i| = 0$$

   if and only if $x_i = y_i$ for all i, that is, if and only if $\underline{x} = \underline{y}$. Otherwise $d_1(\underline{x}, \underline{y}) > 0$.
   ii.

$$d_1(\underline{x}, \underline{y}) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |y_i - x_i| = d_1(\underline{y}, \underline{x}).$$

iii. By the triangle inequality on $\mathbb{R}^1$,

$$d_1(\underline{x},\underline{z}) = \sum_{i=1}^{n} |x_i - z_i| \\ \leq \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) = d_1(\underline{x},\underline{y}) + d_1(\underline{y},\underline{z}).$$

<u>ℓ^∞ metric</u>:

i. $d_\infty(\underline{x},\underline{y}) = \max_i\{|x_i - y_i|\} = 0$ if and only if $|x_i - y_i| = 0$ for all i, which holds if and only if $x_i = y_i$ for all i, that is $\underline{x} = \underline{y}$. Otherwise $d_\infty(\underline{x},\underline{y}) > 0$.

ii. Since $|x_i - y_i| = |y_i - x_i|$, $d_\infty(\underline{x},\underline{y}) = d_\infty(\underline{y},\underline{x})$.

iii. In $d_\infty(\underline{x},\underline{z})$, let $|x_i - z_i|$ obtain the maximum. By the triangle inequality on $\mathbb{R}^1$,

$$d_\infty(\underline{x},\underline{z}) = |x_i - z_i| \\ \leq |x_i - y_i| + |y_i - z_i| \leq d_\infty(\underline{x},\underline{y}) + d_\infty(\underline{y},\underline{z}).$$

The inequality

$$d_\infty(\underline{x},\underline{y}) \leq d_1(\underline{x},\underline{y}) \leq n d_\infty(\underline{x},\underline{y})$$

implies that $\lim_{k\to\infty} d_\infty(\underline{x}_k,\underline{x}) = 0$ if and only if $\lim_{k\to\infty} d_1(\underline{x}_k,\underline{x}) = 0$, so $\{\underline{x}_k\}_{k\in\mathbb{N}}$ converges in d_1 if and only if it converges in d_∞.

Problem 3. The binomial coefficients are defined for integers $0 \le k \le n$ by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

a. (5 points) Decide, with proof, whether the series $\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}}$ converges.

b. (5 points) Prove that $\frac{\binom{2n}{n}}{2^{2n}} \to 0$ as $n \to \infty$.
[Hint: first check that $\frac{\binom{2n}{n}}{2^{2n}} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \dots \cdot \frac{1}{2}$.]

Solution.

a. We check that the series converges by the ratio test. For $n \ge 1$,

$$\frac{\binom{2n}{n}}{\binom{2n+2}{n+1}} = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4} < 1$$

as $n \to \infty$, so that the condition of the ratio test is met.

b. We first check the identity for $\frac{\binom{2n}{n}}{2^{2n}}$ by induction.
Base case ($n = 1$): We have $\frac{\binom{2}{1}}{2^2} = \frac{1}{2}$ as wanted.
Inductive step: Assume for some $n \ge 1$ that

$$\frac{\binom{2n}{n}}{2^{2n}} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \dots \cdot \frac{1}{2}.$$

Then

$$\begin{aligned} \frac{\binom{2n+2}{n+1}}{2^{2n+2}} &= \frac{(2n+2)(2n+1)}{4(n+1)^2} \cdot \frac{\binom{2n}{n}}{2^{2n}} \\ &= \frac{2n+1}{2n+2} \cdot \frac{2n-1}{2n} \cdot \dots \cdot \frac{1}{2}, \end{aligned}$$

completing the inductive step.

Let, for $n \ge 2$,

$$\begin{aligned} s_n &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \dots \cdot \frac{1}{2} = \frac{\binom{2n}{n}}{2^{2n}} \\ t_n &= \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdot \dots \cdot \frac{2}{3}. \end{aligned}$$

Note that the product defining t_n has one fewer term than that defining s_n. By comparing term-by-term,

$$t_n > s_n > \frac{1}{2}t_n.$$

Also, both sequences are bounded below, and decreasing, hence converge to a non-negative limit. Let $s_n \to s$, $t_n \to t$. Then $s_n t_n \to st$. But $s_n t_n$ is a telescoping product, equal to $\frac{1}{2n}$, so $st = 0$. The inequalities imply $t \geq s \geq \frac{t}{2}$ and hence $s = t = 0$.

Problem 4. (10 points) Prove that a continuous function on a closed bounded interval $[a, b]$ is uniformly continuous.

Solution. Suppose for contradiction that f is continuous, but not uniformly continuous on $[a, b]$. Let $\epsilon > 0$ violate the definition of uniform continuity for f. Hence there are sequences of points $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ in $[a, b]$ with $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon$. By the Bolzano-Weierstrass Theorem there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ which converges to $x \in [a, b]$. By the triangle inequality,

$$|x - y_{n_k}| \leq |x - x_{n_k}| + |x_{n_k} - y_{n_k}| \leq |x - x_{n_k}| + \frac{1}{n_k}$$

tends to 0 as $k \to \infty$, so $y_{n_k} \to x$, also. By continuity of f at x, $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$, so $|f(x_{n_k}) - f(y_{n_k})| \to 0$, a contradiction.