MATH 314, SPRING 2026 FINAL EXAM

Each problem is worth 10 points.

Problem 1. The group algebra $\mathbb{C}(S_n)$ of the symmetric group on n letters consists of sum $\sum_{\sigma\in S_n} c_\sigma \sigma$, where c_σ is in $\mathbb{C}$. Prove that $\frac{1}{\binom{n}{2}}\sum_{1\le i<j\le n}(i,j)$ is central in the group algebra. Using this and Schur's lemma, conclude that if R is an irreducible matrix representation of S_n then $\frac{1}{\binom{n}{2}}\sum_{1\le i<j\le n} R((i,j))$ is a constant multiple of the identity and determine the constant in terms of the character.

Problem 2. Consider a matrix with columns indexed by tuples $1 \le i_1, ..., i_k \le n$ distinct. Given a permutation σ of S_n an entry 1 appears at row $\sigma(i_1), ..., \sigma(i_k)$ and all other entries 0. Prove that this defines a representation of the symmetric group, and determine the multiplicity with which the trivial representation of the symmetric group appears in this representation. Using this or otherwise, find a formula for the moments

$$\frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{Fix}(\sigma)^k$$

where $\mathrm{Fix}(\sigma)$ is the number of fixed points of σ.

Problem 3. Determine the dimensions and Lie algebras of the matrix groups $U_n, SU_n, O_{3,1}, SO_n(\mathbb{C})$.

Problem 4. Determine if the primes 2, 3, 11, 17 ramify, split or remain inert in the fields $\mathbb{Q}(\sqrt{-13}), \mathbb{Q}(\sqrt{101})$.

Problem 5. Show that for any integer matrix A there is an invertible matrix P such that AP has Hermitian normal form

$$\begin{pmatrix} d_1 & 0 & \cdots & & \\ a_2 & d_2 & 0 & \cdots & \\ a_3 & b_3 & d_3 & 0 & \cdots \\ \vdots & & & & \end{pmatrix}$$

with non-negative entries and $d_1 > 0$, $a_2 < d_2$, $a_3, b_3 < d_3$,.... Using this describe the orbits when 2×2 integer matrices of determinant p, p prime are acted on by right multiplication by $\mathrm{GL}_2(\mathbb{Z})$.

Problem 6. How many elements of the fields F_{p^n} have order n over F_p? How many degree n polynomials are irreducible over F_p?

Problem 7. Determine the degrees of the splitting fields over $\mathbb{Q}$ of $x^3 - 2$, $x^4 - 1$ and $x^4 + 1$.

Problem 8. Let $s_1, ..., s_n$ be the elementary symmetric functions in variables $u_1, ..., u_n$ and let F be a field. Prove that the field $F(u_1, ..., u_n)$ of rational functions in $u_1, ..., u_n$ is a Galois extension of $F(s_1, ..., s_n)$ with Galois group S_n.