MATH 314, SPRING 2025 PRACTICE FINAL

Each problem is worth 10 points.

Problem 1. Find, with proof, a description of the tangent space at the identity of the group $SL_n(\mathbb{R})$ of $n \times n$ real matrices of determinant 1.

Problem 2. Given a group G, a 1-dimensional character of G is a group homomorphism $\chi : G \to \mathbb{C}$. The dual group of Abelian G is $\hat{G}$, the group of 1-dimensional characters under multiplication of functions. Given a finite abelian group $G = (\mathbb{Z}/d_1\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/d_k\mathbb{Z})$ prove that given integers $b_1, ..., b_k$, there is a character $\chi_b(a_1, ..., a_k) = e^{2\pi i(\frac{a_1 b_1}{d_1} + \cdots + \frac{a_k b_k}{d_k})}$ and that all characters arise this way. Hence or otherwise prove G and $\hat{G}$ are isomorphic.

Problem 3. Give the proof that an odd prime p is the sum of two squares if and only if $p \equiv 1 \bmod 4$.

Problem 4. Given variables $x_1, ..., x_n$ write $\delta(x_1, ..., x_n) = (x_1 - x_2)(x_1 - x_3)...(x_{n-1} - x_n)$. Prove that if $\sigma \in S_n$ permutes the x_i, σ acts on δ by multiplying it by the sign of σ. Hence or otherwise explain that if K/F is the splitting field of a cubic irreducible polynomial f with Galois group S_3 then the discriminant of f is not a square in F.

Problem 5. Give the proof that the multiplicative group of a finite field is cyclic. (Hint: the group is a finite abelian group, use the structure theorem.)

Problem 6. Given an $n \times n$ complex matrix A, its characteristic polynomial is $P_A(x) = \det(xI - A)$. Using Jordan normal form, prove $P_A(A) = 0$.